

Quiz 3

Full Marks: 20, Time: 1 hour

Roll Number: _____ Name: _____

1. Answer each question on a new page of the answer booklet.
2. Do not use pencils. Pens only!
3. The paper has a bonus of 4 marks. If you score above 20, I will add it to your final quiz average.
4. Unless mentioned explicitly, assume that the graph has no self-loops and no vertex with degree 0.

Definitions/Theorems you might need:

- Recall, for functions $f : V \rightarrow \mathbb{R}$, we defined the normalized adjacency operator K as: $(Kf)(u) := \mathbb{E}_{v \sim u}[f(v)]$, where $v \sim u$ refers to picking a random neighbour v of u .
- The normalized Laplacian operator is defined as $L := I - K$, where I is the identity operator.
- *Spectral Theorem:* For a given graph G , $\exists$ orthonormal functions $\mathbb{1} = \phi_0, \phi_1, \dots, \phi_{n-2}, \phi_{n-1} : V \rightarrow \mathbb{R}$ and real numbers $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-2} \leq \lambda_{n-1} \leq 2$ such that $L\phi_i = \lambda_i \phi_i, \forall i$. $\{\phi_0, \phi_1, \dots, \phi_{n-1}\}$ forms an orthonormal basis for $\{f : V \rightarrow \mathbb{R}\}$.
- We also saw that $\phi_0, \dots, \phi_{n-1}$ are also the eigenvectors of K with corresponding eigenvalues: $\kappa_i = 1 - \lambda_i$, for all i .
- *Random Walk Convergence:* Let $|\kappa_i| \leq 1 - \varepsilon$ for all $0 < i \leq n - 1$ ($0 < \varepsilon < 1$). Then, for any worst-case distribution ρ_0 on V , if $u_0 \sim \rho_0$ and $u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_t$ (t -random walk) with $t \geq (c \ln n)/\varepsilon$ for some constant $c > 1$, then ρ_t (the distribution on u_t) is $1/n^{2c-2}$ close (in the chi-squared divergence) to the stationary distribution π .

Problem 1: [4 marks]

Show that a graph G is connected if and only if $\lambda_1 > 0$, where λ_1 is the second smallest eigenvalue of L .

Problem 2: [6 marks (3+3)]

Let $G' = (V, E')$ be a "lazy" version of the graph $G = (V, E)$ with the same vertex set such that each vertex v in G' gets $\deg_G(v)$ self-loops added to it. Therefore, random walk on G' is like that on G , except that now with probability $1/2$ at each step, you lazily stay on the same vertex.

- A) Show that the Laplacian L' of G' is $\frac{1}{2}L$.
- B) Defining $\lambda'_i = \frac{1}{2}\lambda_i$ for all i , show that (λ'_i, ϕ_i) is an eigenvalue/eigenfunction pair for L' , and that $0 \leq \lambda'_i \leq 1$, for all i .

Problem 3: [10 marks (4+3+3)]

Let $G_{\text{path}} = (V, E)$ be a path graph on n vertices, i.e., $E = \{(v, v+1) : 1 \leq v < n\}$.

- A) Find the eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ of G_{path} 's Laplacian L .
- B) Find the eigenspaces (find the basis of each eigenspace) corresponding to each of the eigenvalues above.
- C) Find an asymptotic lower bound for the time t of the convergence of random walk.

Problem 4: [4 marks]

For any two distributions ρ and π , the chi-squared divergence and the total variation distance are defined as:

$$d_{\chi^2}(\rho, \pi) = \mathbb{E}_{u \sim \pi}[(f(u) - 1)^2] \text{ and } d_{TV}(\rho, \pi) = \frac{1}{2} \sum_{u \in V} |\rho(u) - \pi(u)|,$$

respectively. Prove that $d_{TV}(\rho, \pi) \leq \frac{1}{2} \sqrt{d_{\chi^2}(\rho, \pi)}$.