

Question 1: Show that a graph G is connected if and only if $\lambda_1 > 0$, where λ_1 is the second smallest eigenvalue of L .

Solution

Suppose G is disconnected $\Rightarrow$ it has at least two disconnected components G_1 & G_2 . By reordering vertices appropriately, we can write the Laplacian as

$$L = \begin{bmatrix} L_{G_1} & 0 \\ 0 & L_{G_2} \end{bmatrix}$$

corresponding to G_1 (pointing to L_{G_1}) and corresponding to G_2 (pointing to L_{G_2})

$\Rightarrow$ $\exists$ at least two orthogonal eigenvectors of the eigenvalue 0:

$$\Phi_0 = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}_{n \times 1} \quad \& \quad \Phi_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}_{n \times 1}$$

#1's = # vertices in G_1 (pointing to the $\mathbf{1}$ in Φ_0)
 (# vertices of $G = n$)
 #1's = # vertices in G_2 (pointing to the $\mathbf{1}$ in Φ_1)

$$\Rightarrow \lambda_0 = \lambda_1 = 0$$

Conversely, suppose G is connected & ψ is an eigenvector of L corresponding to eigenvalue 0, i.e. $L\psi = 0$

$$0 = \langle \psi, L\psi \rangle = E[\psi] = \frac{1}{2} \sum_{(u,v) \in E} (\psi(u) - \psi(v))^2 \cdot \Pr[u_E = u, v]$$

uniform distribution on E .

$\Rightarrow$ For every pair of vertices (u, v) connected by an edge (i.e. $\Pr[u_E = u, v] \neq 0$), we have $\psi(u) = \psi(v)$.

$\Rightarrow$ By induction, you can show that every pair of vertices (u, v) connected by a path will have $\psi(u) = \psi(v)$.

$\therefore G$ is connected, $\therefore \forall u, v \psi(u) = \psi(v)$.

$\Rightarrow \psi$ is constant & $\dim(\text{Eigenspace corr. to } 0) = 1$.

Question 2: $G' = (V', E')$ be a "lazy" version of graph $G = (V, E)$ with the same vertex set such that each vertex v in G' gets $\deg_G(v)$ self-loops added to it. Therefore, random walk on G' is like that on G , except that now w.p. $\frac{1}{2}$ at each step, you lazily stay on the same vertex.

A) Show that the Laplacian L' of G' is $\frac{1}{2}L$.

B) Defining $\lambda'_i = \frac{1}{2}\lambda_i \neq 0$, show that (λ'_i, ϕ_i) is an eigenvalue/eigenvector pair for L' , and that $0 \leq \lambda'_i \leq 1 \neq 0$.

Solution:

A) Consider the normalized Adjacency matrix: $K' = I - L'$ & $K = I - L$

$K_{uv} = \frac{\text{\#edges between } u \& v}{\deg(u)} \quad \forall u \& v \in V \quad \& \quad K_{uu} = 0 \quad \forall u \in V$
 (no self-loops)

$\downarrow$

(recall this captured the probability that v is chosen as a random neighbor of u .)

Now, $\because G'$ is s.t. $\forall v \in V$, you add $\deg_G(v)$ self-loops

$\therefore$

$$K'_{uu} = \frac{1}{2} \quad \forall u \in V \quad \& \quad K'_{uv} = \frac{1}{2} K_{uv} \quad \forall u \neq v \in V$$

$$\therefore K' = \frac{I}{2} + \frac{K}{2}$$

$$\Rightarrow L' = I - K' = \frac{1}{2}(I - K) = \frac{1}{2} \cdot L$$

$$B) \quad L' \phi_i = \frac{1}{2} \cdot L \phi_i = \frac{1}{2} \cdot \lambda_i \cdot \phi_i = \lambda'_i \cdot \phi_i \quad \forall i$$

Question 3: $G_{\text{path}} = (V, E)$ be path graph on n vertices
i.e. $E = \{(v, v+1) : 1 \leq v < n\}$.

- Find eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ of G_{path} 's Laplacian L .
- Find eigenspace corresponding to each λ_i .
- Find an asymptotic lower bound for time t of the convergence of random walk.

Solution (sketch)

★ The cleanest way to solve this question is to first consider ring graph (makes things symmetric)
Recall, ring graph has all edges of G_{path} + 1 additional edge $(1, n)$.

$$L_{\text{ring}} = \begin{bmatrix} 0 & 1/2 & 0 & \dots & 0 & 1/2 \\ 1/2 & 0 & 1/2 & & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & \dots & 0 \\ \vdots & & & & & \vdots \\ 1/2 & 0 & \dots & & 1/2 & 0 \end{bmatrix}_{n \times n}$$

$$L_{\text{ring}} = \begin{bmatrix} 1 & -1/2 & 0 & \dots & 0 & -1/2 \\ -1/2 & 1 & -1/2 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ -1/2 & 0 & 0 & \dots & -1/2 & 1 \end{bmatrix}_{n \times n}$$

Now, the cute trick is to notice that L_{ring} is such that each row is obtained by "shifting" the previous row (these are generally called circulant matrices). More formally

Define "shift" matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & & 0 \end{bmatrix}_{n \times n}$$

★ Prove that:

$$L_{\text{ring}} = I - \frac{1}{2} \cdot P - \frac{1}{2} \cdot P^{n-1}$$

(easy to find P^{n-1} , then it is obvious why L_{ring} is this value)

Now P is a simpler matrix to deal with.

Now, show that characteristic polynomial of P is $(x^n - 1) \cdot (-1)^n$
 $\therefore$ roots are the n -th roots of unity: $w_0, w_1, \dots, w_{n-1}$ where
 $w_k = e^{2\pi i k / n} = \left(\cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n} \right)$

★ Note
there are direct ways of dealing with paths also but trickier recursions to prove there.

∴ eigenvalues of P^{n-1} will be: $w_0^{n-1}, w_1^{n-1}, \dots, w_{n-1}^{n-1}$

∴ eigenvalues of L_{ring} will be: $\left(1 - \frac{w_k}{2} - \frac{w_k^{n-1}}{2}\right) \forall k=0, \dots, n-1$

$$= \frac{1}{2} (2 - (w_k + w_k^{n-1})) = \frac{1}{2} (2 - (\cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n} + \cos \frac{2\pi k}{n} - i \sin \frac{2\pi k}{n}))$$

$$= \frac{1}{2} (2 - 2 \cdot \cos \frac{2\pi k}{n}) = \left(1 - \cos \frac{2\pi k}{n}\right)$$

∴ Eigenvalues of L_{ring} are: $\left(1 - \cos \frac{2\pi k}{n}\right) \forall k=0, 1, \dots, n-1$

Eigenvectors:

- Prove ϕ_k is an eigenvector of P corresponding to w_k iff ϕ_k is an eigenvector of L_{ring} corresponding to $\left(1 - \cos \frac{2\pi k}{n}\right) \forall k$.

∴ Task reduces to finding eigenvectors of P . Show that the eigenspace corresponding to w_k is of dimension 1, generated by

$$\phi_k = \begin{pmatrix} 1 \\ w_k \\ \vdots \\ w_k^{n-2} \\ w_k^{n-1} \end{pmatrix}, \quad \forall k=0, 1, \dots, n-1$$

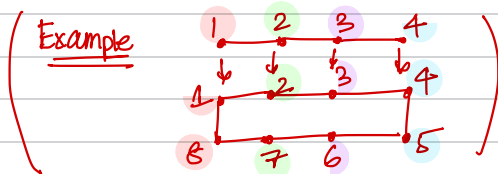
⇒ ϕ_k is also eigenvector for L_{ring} corresponding to $\left(1 - \cos \frac{2\pi k}{n}\right) \forall k=0, 1, \dots, n-1$.

Coming back to G_{path} with n -vertices. We will map each vertex $v \in V_{path} = [n]$ to 2 vertices in the ring graph G_{ring} with $2n$ -vertices,

$$v \in V_{path} = [n] \rightarrow v \in V_{ring} = [2n]$$

$$\rightarrow (2n+1) - v \in V_{ring} = [2n]$$

(called as quotient of G_{ring})



$$L_{ring_{2n}} = \begin{pmatrix} L_{path_n} & H_n \\ H_n & L_{path_n} \end{pmatrix}_{2n \times 2n} \text{ where } H_n = \begin{pmatrix} 0 & 0 & \dots & 0 & -1/2 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ -1/2 & 0 & \dots & 0 & 0 \end{pmatrix}_{n \times n}$$

Now $\forall k = 0, \dots, 2n-1$, $\lambda_k = \left(1 - \cos \frac{\pi k}{n}\right)$ is eigenvalue of $L_{ring_{2n}}$ with corresponding eigenvector $\phi_k = (1 \ w_k \dots \ w_k^{2n-1})^T$.

$$\Rightarrow L_{ring_{2n}} \phi_k = \lambda_k \phi_k \quad \forall k = 0, \dots, 2n-1$$

$$\Rightarrow L_{path_n} \begin{pmatrix} 1 \\ w_k \\ \vdots \\ w_k^{n-1} \end{pmatrix} = \lambda_k \begin{pmatrix} 1 \\ w_k \\ \vdots \\ w_k^{n-1} \end{pmatrix} \quad \forall k = 0, \dots, 2n-1$$

(first n are distinct, remaining are just repeated values)

$\therefore$ eigenvalues of L_{path_n} are $\lambda_k = \left(1 - \cos \frac{\pi k}{n}\right) \quad \forall k = 0, \dots, n-1$

with the corresponding eigenvectors: $\phi_k^1 = (1 \ w_k \dots \ w_k^{n-1})^T$

c) $\rightarrow$ simple now (just plug-in the convergence theorem & asymptotically bound to, based on the bound for the λ_i^1 's for L_{path} found above)

****** Not solving Question 4 as it is simple use of just the definition & very simple inequalities $\rightarrow$ try it yourself if you could not solve it during the quiz.