

Lecture 13

Instructor: Sruthi Sekar

Scribe: Pranav Kandwal

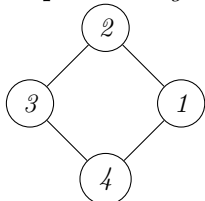
1 Graphs and its Properties

Definition 1 (Graphs). Graph is defined by a tuple (V, E) , where V is the set of vertices and E is the set of edges. $E \subset \{(a, b) \mid a, b \in V\}$. Typically, we'll look at undirected graphs, i.e., both the edges, (a, b) and (b, a) are included.

Example 2. Path $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$

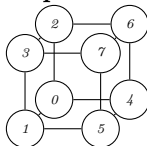
Here $V = \{1, 2, \dots, n-1, n\}$ and $E = \{(i, i+1) \mid i \in [n-1]\}$

Example 3. Ring



Here $V = \{1, 2, \dots, n-1, n\}$ and $E = \{(i, i+1) \mid i \in [n-1]\} \cup \{(1, n)\}$. The above is an example of a ring with $n = 4$.

Example 4. Hyperbole



Here $V = \{0, 1\}^k$ and $E = \{(u, v) \mid u, v \in V \text{ and } \sum_{i=1}^k |u_i - v_i| = 1\}$. Above is an example of hyperbole with $k=3$.

1.1 Certain Assumptions

- Finite Graphs, i.e., $|V|$ and $|E|$ are finite
- Multiple parallel edges allowed
- Self-loops are not allowed
- Do not allow vertices with degree 0

Definition 5 (Degree of a vertex). $\deg(v)$ is defined as the total number of edges incident on that vertex. A graph is d -regular if every vertex has the same degree d . For example, hyperbole is a k -regular graph.

2 Labeling of vertices

Definition 6. Let $f : V \rightarrow \mathbb{R}$ be the function to label vertices in the graph by real numbers. Examples: f is temperature(graph models physical objects), f is voltage(graph models electrical network).

$f : V \rightarrow \mathbb{R}$ can also be represented as a column vector.

$$\mathbf{f}(\mathbf{v}) = \begin{bmatrix} f(v_1) \\ f(v_2) \\ | \\ f(v_n) \end{bmatrix}$$

Moreover, $\{f : V \rightarrow \mathbb{R}\}$ forms a vector space over $\mathbb{R}$ of dimension $|V|$.

Example 7. Now we will look at a concrete example of such a function f . $f = 1_S$ be a 0,1 indicator of a subset S of vertices. Let $S \subset V$, then

$$f(v) = \begin{cases} 1, & \text{if } v \in S \\ 0, & \text{if } v \notin S \end{cases}$$

3 Dirichlet Form/Local Variance($\mathcal{E}$)

Definition 8. Informally, it is defined as the average squared value of function values along the edges. Sometimes, it is also referred as energy function. More formally,

$$\mathcal{E}(f) = \frac{1}{2} [\mathbb{E}_{(u,v)} [(f(u) - f(v))^2]]$$

where (u, v) means uniform random edge $(u,v) \in E$

Example 9. Let $S \subset V$ and $f = 1_S$, then

$$\mathcal{E}(f) = \frac{1}{2} \mathbb{E}_{(u,v)} [(1_S(u) - 1_S(v))^2] = \frac{1}{2} (\text{fraction of edges on } \delta S),$$

where δS is the boundary of S

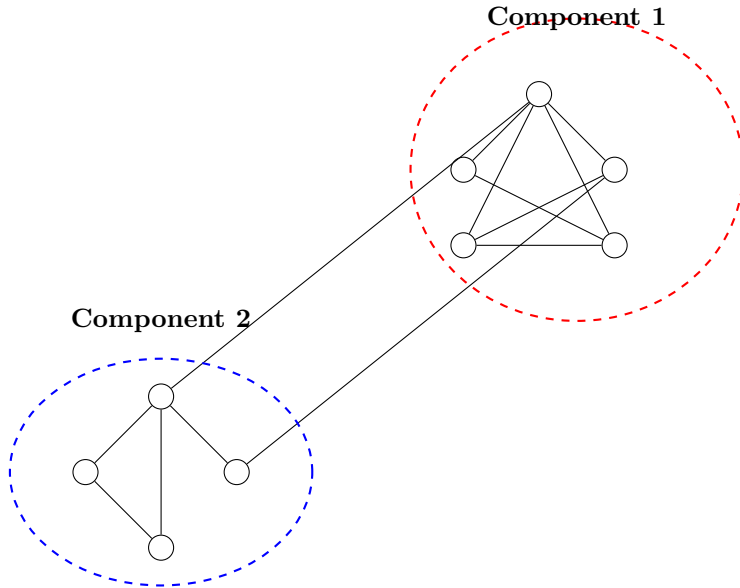
3.1 Some Properties of Dirichlet Form

- $\mathcal{E}(f) \geq 0$
- $\mathcal{E}(f + c) = \mathcal{E}(f)$
- $\mathcal{E}(cf) = c^2 \mathcal{E}(f), c \in \mathbb{R}$

3.2 Nature of $\mathcal{E}$

Intuition. $\mathcal{E}(f) \iff$ f values don't vary "much" along the edges.

Example to understand the intuition. Consider the "almost-disconnected" graph.



Very few edges across S and S^c , i.e., very few edges on δS . Recall from our previous discussion, this is captured by $\mathcal{E}(1_S)$. So if $\mathcal{E}(1_S)$ is small $\implies$ prevents fast convergence to limiting distributions. Consider d -regular graph for better understanding. Informally, the statement says that, if there are a few edges on δS , it will block the random walk to getting to uniform distribution on all vertices in “short” time! Moreover, in the subsequent lectures we will see that: “Fast convergence ” $\iff \mathcal{E}(f)$ “never small”. We have to take help from Linear Algebra to formalize this!!

4 Random Vertices and Random Walks

If we pick $v \xleftarrow{\$} V$, then the resulting distribution on V (which is uniform) does not give us any useful information about the vertex (eg. degree information). So, the correct way to pick a “random” vertex is:

- Choose a random edge $e(u, v) \xleftarrow{\$} E$.
- Output u .

We call the resulting distribution π . Hence $\pi(v) = \frac{\deg(v)}{2|E|}$. Note that for a d -regular graph π is a uniform distribution.

Example 10. Consider the given graph Figure 1 and the π distribution over it. Note that for an edge between the the vertices u and v , we could have picked (u, v) or (v, u) but output u only in the first case, hence the factor of $\frac{1}{2}$.

Remark.

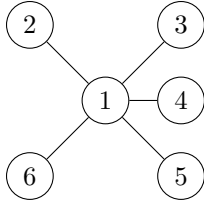
- Pick $u \sim \pi$. u is picked from V through π .
- Then pick v as a uniform random neighbor of u .

The distribution of v is also π .

Definition 11 (Random Walk). Consider a random walk of t steps starting at $u_0 \in V$. This means that at each step, you pick a random neighbor of the vertex you are currently at.

Proposition: Let $t \in \mathbb{N}$, pick $u \sim \pi$. Now take a random walk starting at u and taking t steps.

Figure 1: π distribution



v	$\pi(v)$
1	$\frac{1}{2}$
2	$\frac{1}{10}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{10}$

$$u \xrightarrow{1} v_1 \xrightarrow{1} v_2 \dots v_{t-1} \xrightarrow{1} v$$

Then the distribution of v is π , i.e., $v \sim \pi$

Definition 12 (Invariant/Stationary Distribution). π is the stationary distribution on V

Question? If $u_0 \in V$ is not chosen from π and we start a random walk at u_0 , then as $t \rightarrow \infty$, does the distribution of $v \rightarrow \pi$?

- No, if G is disconnected.
- No, if G is bipartite.
- Otherwise, yes!

Question? How large should t be to get ε -close to π by doing a random walk? This we will cover in the subsequent lectures!!

Definition 13. Let $f : V \rightarrow \mathbb{R}, u \sim \pi$.

- mean(μ): $\mathbb{E}[f] = \mathbb{E}_{u \sim \pi}[f(u)]$
- variance: $Var(f) = \mathbb{E}_{u \sim \pi}[(f(u) - \mu)^2] = \mathbb{E}_{u \sim \pi}[f(u)^2] - \mathbb{E}_{u \sim \pi}[f(u)]^2$

Example 14. • Let $f = 1_S$, where $S \subset V$.

- $\mu = \mathbb{E}[1_S] = Pr_{u \sim \pi}[u \in S] = vol(S)$.
- $Var(f) = \mathbb{E}[(1_S)^2] - \mu^2 = \mu - \mu^2 = \mu(1 - \mu) = vol(S) - vol(S)^2$

5 Exercises

Ex1. Show that $\mathcal{V} = \{f : V \rightarrow \mathbb{R}\}$ with $|V| = n$ is an n -dimensional vector space (with the vector addition and scalar multiplication over $\mathbb{R}$).

Sol.1) Let us consider two functions $f, g \in \{f : V \rightarrow \mathbb{R}\}$. Then,

- $(f + g)(v) = f(v) + g(v)$
 $= h(v) \in \mathbb{R}$, since $f(v)$ and $g(v) \in \mathbb{R}$.
Hence we get a new labeling of vertex v , i.e., $h(v) \in \mathcal{V}$, such that, $h(v) = f(v) + g(v) \forall v \in V$.
- Since, $c * f(v) \in \mathbb{R}$, hence we get a new labeling $f'(v) = c * f(v), \forall v \in V$ and for $c \in \mathbb{R}$

Hence, $\mathcal{V}$ is closed under vector addition and scalar multiplication.

Ex2. Show that $\pi(v) = \frac{\deg(v)}{2|E|}$.

Sol. 2) Recall, how we defined the distribution π . We first choose a random edge $e(u, v) \xleftarrow{\$} E$ and output u . $Pr[e(u, v) \xleftarrow{\$} E] = \frac{1}{2|E|}$. Next, $Pr[\text{output } u | \text{edge}(u, v) \in E] = \frac{1}{2}$. This is because, $e_{v \sim u} \in E$ can be viewed as a directed pair (u, v) and (v, u) each occurring with equal probability. Finally, the total probability,

$$Pr[\text{output } u] = \sum_{\text{edges } e \text{ incident on } u} \left(Pr[e(u, v) \xleftarrow{\$} E] \cdot Pr[\text{output } u | \text{edge}(u, v) \in E] \right)$$

$$Pr[\text{output } u] = \frac{1}{2|E|} \sum_{\text{edges } e \text{ incident on } u}$$

$$Pr[\text{output } u] = \frac{1}{2|E|} \cdot \deg(u) \quad \because \text{def}^n \text{ of } \deg(u)$$

Hence Proved!

Ex3.

- Pick $u \sim \pi$.
- Let v be a uniform random neighbor of u .

Show that the above is equivalent to “Picking a uniform random edge $e(u, v) \in E$ ”.

Sol. 3)

$$Pr[e(u, v) \xleftarrow{\$} E] = \frac{1}{|E|} \tag{1}$$

But each edge $e_{v \sim u} \in E$ is unordered, i.e., it can be seen as both (u, v) and (v, u) and both the ordering is equally likely, hence

$$Pr[e(u, v) \xleftarrow{\$} E] = \frac{1}{2|E|} \tag{2}$$

Now let us consider the method above and try to compute the probability of selecting the edge $(e(u, v) \in E)$ using it.

$$Pr[u \sim \pi] = \pi(u) \text{ or } \frac{\deg(u)}{2|E|} \tag{3}$$

$$Pr[v \sim \text{neighbor}(u)] = \frac{1}{\deg(u)} \tag{4}$$

Hence using (3) and (4), required probability: $Pr = \frac{\deg(u)}{2|E|} * \frac{1}{\deg(u)} = \frac{1}{2|E|}$

Hence Proved!

Ex4. Show that $Var(f) = \frac{1}{2} \mathbb{E}_{u, v \sim \pi} [(f(u) - f(v))^2]$. This term is also known as the global variance.

Sol. 4)

$$\frac{1}{2} \mathbb{E}_{u, v \sim \pi} [(f(u) - f(v))^2] = \frac{1}{2} (\mathbb{E}_{u, v \sim \pi} [(f(u)^2 + f(v)^2 - 2f(u)f(v))])$$

$$\frac{1}{2} (\mathbb{E}_{u \sim \pi} [f(u)^2] - 2\mathbb{E}_{u, v \sim \pi} [f(u)f(v)] + \mathbb{E}_{v \sim \pi} [f(v)^2])$$

$$\mathbb{E}_{u \sim \pi} [f(u)^2] - \mathbb{E}_{u, v \sim \pi} [f(u)f(v)] \quad \because \mathbb{E}[f(u)^2] = \mathbb{E}[f(v)^2] \text{ as } u, v \sim \pi$$

$$\mathbb{E}_{u \sim \pi} [f(u)^2] - \mathbb{E}_{u \sim \pi} [f(u)]^2 \quad \because u \text{ and } v \text{ have same distribution}$$

Hence Proved!