

Lecture 3

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1 Factorial and Binomial Coefficients

1.1 Lecture Overview

The purpose of the lecture is to discuss

- Stirling's approximation for $n!$
- Approximation to Binomial Coefficients

The lecture begins with a brief history on how Stirling came up with the formula to approximate $n!$. We then discuss a simple problem that uses the binomial random variable. Then for most of the lecture we try to come up with a function $f(n)$ such that $f(n) \sim n!$. We also talked about a few applications of Stirling's formula.

1.2 Motivation

Exercise 1. *What is the probability that flipping a fair coin $2n$ times, results in exactly n heads and n tails?*

The above problem can be captured by Binomial distribution as each coin toss has only two possible outcomes: heads or tails which we can define as success or failure. We define Binomial Random Variable to help us solve the problem

Definition 2. Binomial Random Variable

A random variable $X = \text{Bin}(n, p)$ where $n \in \mathbf{N}$ taking values $0, 1, 2, \dots, n$ such that

$$\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}, 0 < p < 1 \quad (1)$$

where $\Pr[X = k]$ is the probability of k success in n independent trials and p is the success probability of one trial.

Using the previous definition the required probability is

$$\begin{aligned} \Pr[n \text{ heads in } 2n \text{ trials}] &= \binom{2n}{n} \frac{1}{2}^n \frac{1}{2}^{2n-n} \\ &= \binom{2n}{n} \frac{1}{2}^{2n} \\ &= \frac{(2n)!}{n!n!} \cdot \frac{1}{2^{2n}} \end{aligned}$$

It is difficult for us to see how $n!$ grows for large n . To solve this problem, we need a way to approximate $n!$. In this lecture our main motivation is to approximate $n!$, which will be close to Stirling's approximation.

1.3 Brief History

We talked about the brief history of Stirling's formula. It occurred during correspondence between Stirling and De Moivre in 1720. De Moivre was working on approximating a binomial distribution by a normal distribution. What De Moivre showed in his work is

$$\binom{2n}{n} \frac{1}{2^{2n}} \approx 2.168 \frac{(1 - 1/2n)^{2n}}{\sqrt{2n - 1}} \quad (2)$$

2.168 here is an approximation to a constant which De Moivre could only express as an infinite series. Let us examine the expression $(1 - 1/2n)^{2n}$. We will show that

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n}\right)^{2n} = \frac{1}{e} \quad (3)$$

Exercise 3.

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n}\right)^{2n} = \frac{1}{e}$$

Let $A_n = \left(1 - \frac{1}{2n}\right)^{2n}$ and we need to find $\lim_{n \rightarrow \infty} A_n$. Let us look at $\ln A_n$

$$\begin{aligned} \ln A_n &= 2n \ln \left(1 - \frac{1}{2n}\right) \\ &= \frac{\ln \left(1 - \frac{1}{2n}\right)}{1/2n} \end{aligned}$$

Applying L'Hospital's rule ($\frac{0}{0}$ case)

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln A_n &= \lim_{n \rightarrow \infty} \frac{\frac{1}{2n^2}}{\frac{-1}{2n^2}} \\ &= -1 \end{aligned}$$

$$\lim_{n \rightarrow \infty} A_n = e^{-1} = \frac{1}{e}$$

So we can write $(1 - 1/2n)^{2n} \approx 1/e$ and $\sqrt{2n - 1} = \sqrt{2n(1 - 1/2n)} \approx \sqrt{2n}$. This makes De Moivre's approximation

$$\binom{2n}{n} \frac{1}{2^{2n}} \approx \frac{2.168}{\sqrt{2e}\sqrt{n}} \quad (4)$$

Stirling actually found the true value of 2.1668 to be $e\sqrt{\frac{2}{\pi}}$, making the approximation $\approx \frac{1}{\sqrt{\pi n}}$. This is the first time π appears in the approximation. Stirling extended the definition of $m!$ to non integer values of m by interpolation. Tabulating $\ln m!$ for $m = 6, 7, \dots, 17$, he determined $\ln 10.5!$ by interpolation and by using the functional argument of $m! = m(m-1)!$. He observes that the first ten decimal places of $-1/2!$ agree with $\sqrt{\pi}$.

De Moivre formally proved in 1730 that

$$\binom{2n}{n} \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}} \quad (5)$$

The goal for today is to find $f(n)$ such that $n! \sim f(n)$ i.e

$$\lim_{n \rightarrow \infty} \frac{n!}{f(n)} = 1$$

i.e

$$n! = f(n) \pm o(f(n)) = f(n)(1 \pm o(1))$$

1.4 Approximating $n!$

We took few different ways to approximate $n!$.

1.4.1 Attempt-1 Naive Attempt

Let us try simple upper and lower bounds

$$\begin{aligned} n! &= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \\ &\leq n \cdot n \cdot n \cdots n \cdot n = n^n \end{aligned} \tag{6}$$

Again to lower bound we drop the lower half of the terms and lower bound the rest of the terms

$$\begin{aligned} n! &= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \\ &\geq \frac{n}{2} \cdot \frac{n}{2} \cdot \frac{n}{2} \cdots \frac{n}{2} \cdot \frac{n}{2} \quad (n/2 \text{ times}) = \left(\frac{n}{2}\right)^{n/2} \end{aligned} \tag{7}$$

Let us take logs for both (6) and (7) to better understand the large expressions

$$\frac{n}{2} \log_2 \frac{n}{2} \leq \log_2 (n!) \leq n \log_2 n$$

Now

$$\frac{n}{2} \log_2 \frac{n}{2} = \frac{n}{2} \log_2 n - n \cdot \frac{1}{2} \log_2 2 = \frac{n}{2} \log_2 n - \Theta(n)$$

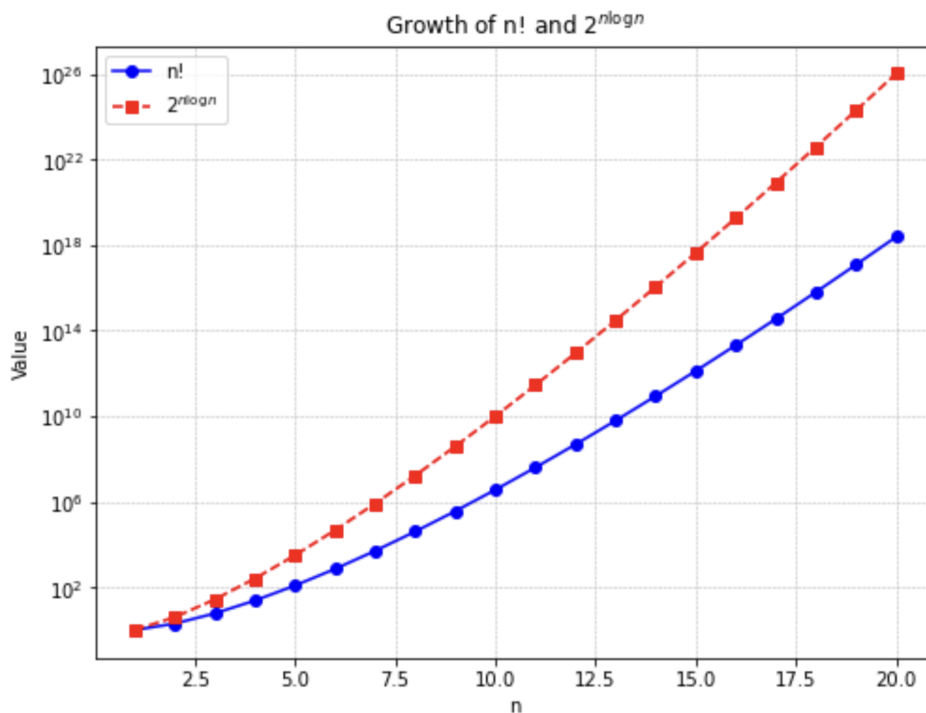
So we get

$$\frac{1}{2} n \log_2 n - \Theta(n) \leq \log_2 (n!) \leq n \log_2 n \tag{8}$$

Exponent to get

$$n! = 2^{\Theta(n \log n)} \tag{9}$$

Let us look at both the graphs



This is a good start, but we can do better.

1.4.2 Attempt-2 using Taylor's Series

Taylor series of e^x is

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ &\geq \frac{x^n}{n!} \quad \forall x \in \mathbf{R}^+ \quad [\text{by dropping all other terms}] \\ \implies n! &\geq \frac{x^n}{e^x} \quad \forall x \in \mathbf{R}^+ \end{aligned}$$

Now we will maximize the right hand side expression to find the tightest lower bound of $n!$. To find maxima

$$\begin{aligned} \frac{d}{dx} \left(\frac{x^n}{e^x} \right) &= 0 \\ \implies \frac{e^x \cdot (nx^{n-1}) - x^n \cdot e^x}{e^{2x}} &= 0 \\ \implies x^{n-1}(n - x) &= 0 \\ \implies x &= n \end{aligned}$$

So the maxima is at $x = n$. Then

$$n! \geq \left(\frac{n}{e}\right)^n \tag{10}$$

Taking $\ln$ both the sides

$$\begin{aligned} \ln n! &\geq n \ln \frac{n}{e} \\ &= n \ln n - n \\ &= n \ln n + \text{error terms} \end{aligned}$$

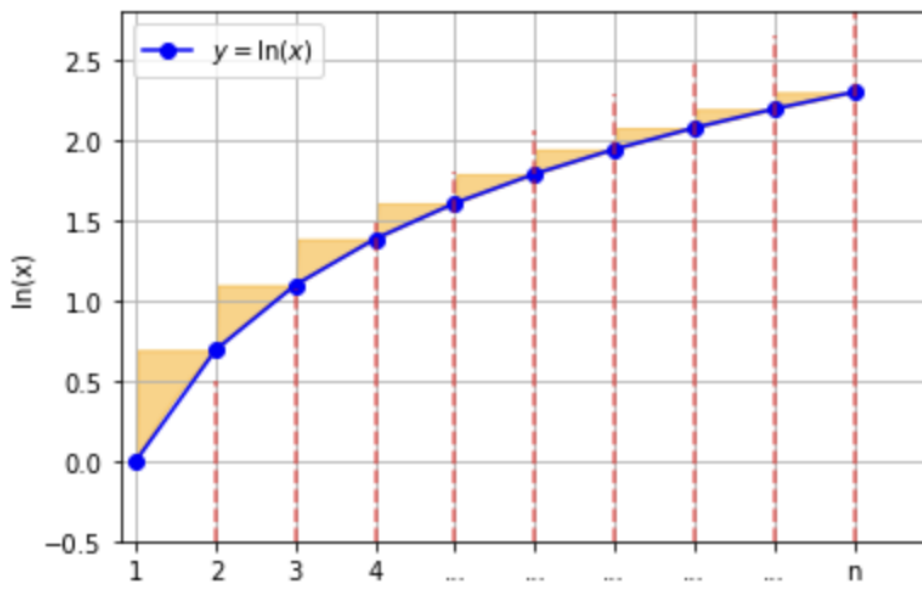
From the previous attempt we already had $\ln n! \leq n \ln n$. So, from the results we can finally write that

$$\ln n! \sim n \ln n \tag{11}$$

We are now interested to find even tighter bound.

1.4.3 Tighter bounds for $n!$ using Integral Trick

$$\ln n! = \ln 1 + \ln 2 + \cdots + \ln n$$



We have plotted the graph and by taking the area of the curve into consideration, we can write

$$\begin{aligned} \ln n! &= \text{area of the rectangles above} \\ &\geq \int_1^n \ln x dx \\ &= n \ln n - n + 1 \end{aligned}$$

Again we can take the upper bound as

$$\ln n! = \int_1^n \ln x dx + \text{the orange colored shaded region}$$

Now if we take the small shaded regions and straighten them as triangles and stack them over each other we get that

$$\begin{aligned} \ln n! &\leq \int_1^n \ln x dx + \frac{1}{2} \ln n \\ &= n \ln n - n + 1 + \frac{\ln n}{2} \end{aligned}$$

So, both the inequalities imply

$$\boxed{\frac{n^n}{e^n} \cdot e \leq n! \leq \frac{n^n}{e^n} \sqrt{n} \cdot e} \quad (12)$$

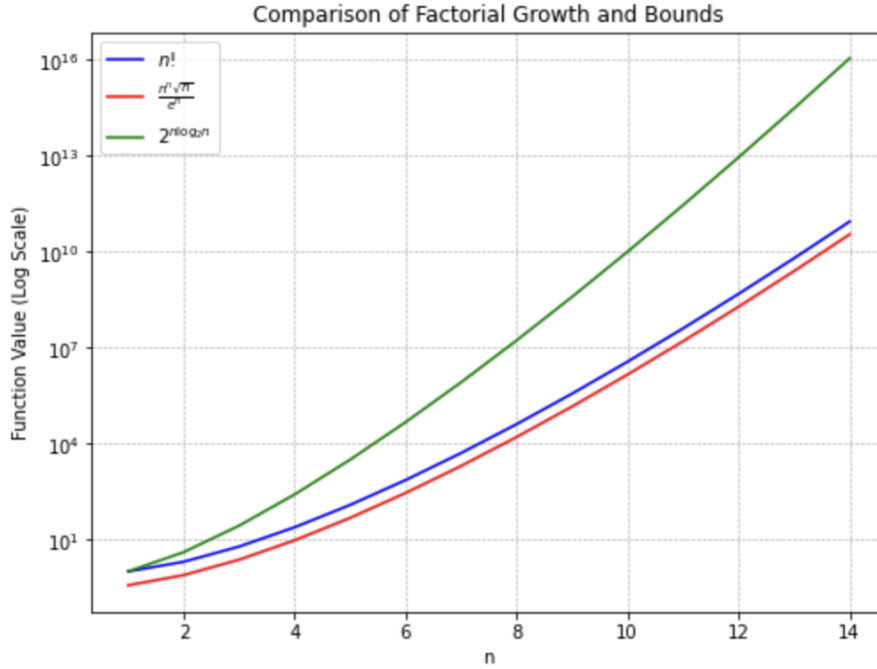
$$\implies n! = \tilde{\Theta}\left(\frac{n^n}{e^n}\right)$$

Exercise 4. Show that while straightening the extra areas to triangle the small sliver for which we did not take into account, has area $\mathcal{O}(1)$ [Proof will be provided later]

So, this gives an even better lower bound i.e

$$\ln n! \geq n \ln n - n + \frac{\ln n}{2} - \mathcal{O}(1)$$

$$\Rightarrow n! = \tilde{\Theta}\left(\frac{n^n}{e^n} \cdot \sqrt{n}\right) \quad (13)$$



The Stirling's formula gives us

$$n! \sim \sqrt{2\pi n} \cdot \frac{n^n}{e^n}$$

Example 5. $10! = 3628800$ and by using Stirling's approximation $\sqrt{2\pi \cdot 10} \cdot \frac{10^{10}}{e^{10}} = 3598695.61$ which is less than 1% error.

1.5 Applications of Stirling's Formula

1.5.1 Binomial Coefficients

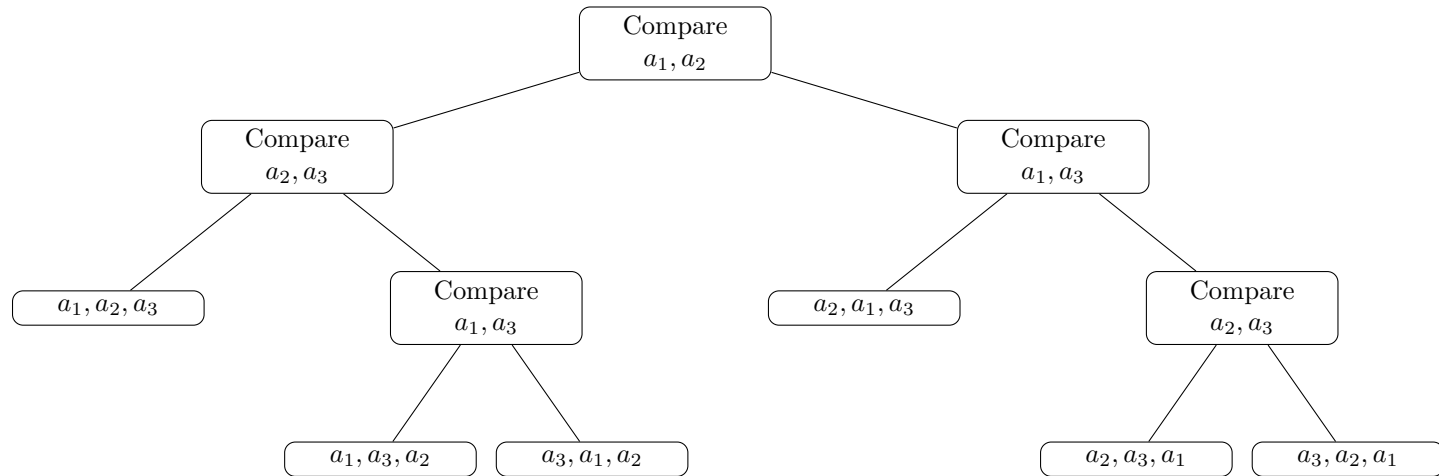
Let us recall that in the first exercise we calculated the probability of flipping a fair coin $2n$ times, resulting exactly n heads and n tails as $\frac{2n!}{n!n!} \cdot \frac{1}{2^{2n}}$. We can use Stirling's here to approximate

$$\begin{aligned} \frac{2n!}{n!n!} \cdot \frac{1}{2^{2n}} &\sim \frac{(2n)^{2n}/e^{2n}}{(n^n/e^n)^2} \cdot \frac{\sqrt{2\pi \cdot 2n}}{2\pi n} \cdot \frac{1}{2^{2n}} \\ &= \frac{2^{2n}}{\sqrt{\pi n}} \cdot \frac{1}{2^{2n}} \\ &= \frac{1}{\sqrt{\pi n}} \end{aligned}$$

Now we want to estimate any $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. There is a case-by-case analysis based on the value of k . We care only for $k = \Theta(n)$. Now, since $\binom{n}{k} = \binom{n}{n-k}$, it suffices to consider $k \leq n/2$. Let $k = pn$ for $0 < p \leq n/2$. Let $q = 1 - p$

$$\begin{aligned} \binom{n}{pn} &= \frac{n!}{(pn)!(qn)!} \\ &\sim \frac{\sqrt{2n\pi}(n/e)^n}{\sqrt{2pn\pi}(pn/e)^{np} \sqrt{2qn\pi}(qn/e)^{nq}} \\ &= \frac{1}{\sqrt{2pq\pi}} \frac{1}{\sqrt{n}} [(1/p)^p (1/q)^q]^n \end{aligned}$$

1.5.2 Minimum Number of Comparisons to Sort



A decision tree to sort three numbers looks like this. At each step, we compare two numbers and there are two branches of that decision for if $a > b$? or $b > a$?. So, a comparison sort is actually a binary tree with $n!$ number of possible outputs i.e $n!$ number of leaves of the binary tree. So, the height of that tree will be $\log_2 n!$.

$\Rightarrow$ minimum number of comparisons needed $= \log_2(n!)$ We will now use Stirling's

$$\begin{aligned} \log_2(n!) &\sim n \log_2(n/e) + \log_2(\sqrt{2n\pi}) \\ &= \Omega(n \log_2(n)) \end{aligned}$$

References