

Lecture 14 (Spectral Graph theory lecture 2)

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1 Recap

First, we do a recap from previous lecture.

Let $G = (V, E)$ be an undirected graph. We are interested in the space of functions that label vertices $\{f : V \rightarrow \mathbb{R}\}$ as an n -dimensional inner product space, where the inner product is defined as $\langle f, g \rangle := \mathbb{E}_{u \sim \pi}[f(u) \cdot g(u)]$. This definition is similar to the dot product but weighted.

Drawing a vertex $u \sim \pi$ means first choosing an edge uniformly at random $u \sim v$ and then outputting u . Recall that $\pi[u]$ is proportional to $\deg(u)$, or more precisely, $\pi[u] = \frac{\deg(u)}{2|E|}$. Therefore, if G is a regular graph, then π is the uniform distribution.

The distribution π is called invariant or stationary because if you pick a random vertex and you take a step in a random walk, then the distribution after the step is also π . In other words, if we select $u_0 \sim \pi$ and take a standard random walk $u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_t$, then each $u_i \sim \pi$.

Our main focus is to study the quantity

$$\mathcal{E}[f] = \frac{1}{2} \mathbb{E}_{u \sim v}[(f(u) - f(v))^2],$$

called Dirichlet form. Informally we call it “local variance” or “energy”.

An interesting case is when $f = 1_S$ for some $S \subseteq V$. Then $\mathcal{E}[f] = \Pr_{u \sim v}[u \in S, v \notin S]$ measures the boundary size of S , that is, the volume of edges going out of S .

Also recall that we can write global variance $\text{Var}_\pi[f]$ as $\frac{1}{2} \mathbb{E}_{u \sim \pi, v \sim \pi}[(f(u) - f(v))^2]$ (here u and v are independent).

2 Limits on $\mathcal{E}[f]$

Now, we ask the question as to which function f minimizes or maximizes $\mathcal{E}[f]$?

Definition 1 (Inner product:). Let $f, g : V \rightarrow \mathbb{R}$ we define $\langle f, g \rangle_\pi = \mathbb{E}_{u \sim \pi}[f(u)g(u)]$

Proposition: $\mathcal{E}[f] = 0$ if and only if f is constant on all connected components of G .

Proof. All the terms in $\mathcal{E}[f]$ must be zero since all the terms are non-negative. So, every edge must have the same value on both ends. Therefore any path must have the same value along the path. In a connected component, every two points are connected by a path. So, every connected component must have f constant. Clearly, if f is constant on all connected components, then it is constant on every edge. Hence $\mathcal{E}[f] = 0$.

Proposition: The number of connected components is the same as the independent f 's with $\mathcal{E}[f] = 0$.

Proof. If $S_1, S_2, \dots, S_\ell$ are connected components then $1_{S_1}, 1_{S_2}, \dots, 1_{S_\ell}$ are linearly independent. The number of linearly independent f which make $\mathcal{E}[f]$ small corresponds to clusters.

Proposition: For all G and for all f , then $\mathcal{E}[f] \leq 2\|f\|_2^2$.

Proof. The proof is just simply manipulation:

$$\frac{1}{2} \mathbb{E}_{u \sim v}[(f(u) - f(v))^2] = \frac{1}{2} \mathbb{E}[f(u)^2] - \mathbb{E}[f(u)f(v)] + \frac{1}{2} \mathbb{E}[f(v)^2] = \mathbb{E}[f^2] - \mathbb{E}[f(u)f(v)]$$

$$\leq \mathbb{E}[f^2] + \sqrt{\mathbb{E}[f(u)^2]\mathbb{E}[f(v)^2]} = 2\mathbb{E}[f^2]$$

where $\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$, which follows from Cauchy-Schwarz.

Exercise 2. Equality is possible if and only if G is bipartite.

3 Normalized adjacency operator

Recall that $\mathcal{E}[f] = \frac{1}{2}\mathbb{E}_{u \sim v}[(f(u) - f(v))^2] = \|f\|_2^2 - \mathbb{E}_{u \sim v}[f(u)f(v)]$. Now let us take a closer look at $\mathbb{E}_{u \sim v}[f(u)f(v)]$. We have that

$$\mathbb{E}_{u \sim v}[f(u)f(v)] = \mathbb{E}_{u \sim \pi}[f(u)\mathbb{E}_{v \sim u}[f(v)]],$$

that is, we pick u and then select a random neighbor v of u . Note that $\mathbb{E}_{v \sim u}[f(v)]$ is the average value of f among the neighbors of u .

Let us call Kf the function that maps u into $\mathbb{E}_{v \sim u}[f(v)]$.

Definition: Given $f : V \rightarrow \mathbb{R}$, define $Kf : V \rightarrow \mathbb{R}$ such that

$$(Kf)(u) := \mathbb{E}_{v \sim u}[f(v)].$$

The operator K is a normalized adjacency operator, Markov operator, or transition operator.

Observe that K is a linear operator, that is, $K(f + g) = (Kf) + (Kg)$. This is true since for any u , $\mathbb{E}_{v \sim u}[(f + g)(v)] = \mathbb{E}_{v \sim u}[f(v) + g(v)] = Kf(u) + Kg(u)$. Just as we can think of function f as a vector, we can think of K as a matrix. Thus, we can think of Kf as a multiplication between a matrix and a vector.

In this view, in order to satisfy $Kf(u) = \mathbb{E}_{v \sim u}[f(v)]$, we must let $K_{uv} = \frac{\# \text{ edges between } u, v}{\deg(u)}$. In other words, K is the adjacency matrix A_G of a graph G but normalized so that the row sums are one. In particular, if G is d -regular, then K is the symmetric matrix $\frac{1}{d}A_G$.

Fact: Row-sum = 1 i.e. K is a stochastic matrix.

Fact: For $f, g : V \rightarrow \mathbb{R}$, $\langle f, Kg \rangle = \mathbb{E}_{u \sim v}[f(u) \cdot g(v)]$.

Proof. By definition,

$$\langle f, Kg \rangle = \mathbb{E}_{u \sim v}[f(u) \cdot (Kg)(u)] = \mathbb{E}_{u \sim \pi}[f(u) \cdot \mathbb{E}_{v \sim u}[g(v)]] = \mathbb{E}_{u \sim v}[f(u) \cdot g(v)].$$

This fact allows us to reach another conclusion. Drawing u and v is the same as drawing v and u and reversing them, so $\langle f, Kg \rangle = \mathbb{E}_{u \sim v}[f(u)g(v)] = \mathbb{E}_{u \sim v}[f(v)g(u)] = \langle g, Kf \rangle = \langle Kf, g \rangle$. That is,

$$\langle f, Kg \rangle = \langle Kf, g \rangle.$$

We say that K is self-adjoint.

Let us put $\langle f, Kg \rangle$ into context with an example. Consider two sets of vertices $S, T \subseteq V$ and let $f = 1_S$ and $g = 1_T$. Then $\langle f, Kg \rangle = \mathbb{E}_{u \sim v}[1_S \cdot 1_T] = \Pr_{u \sim v}[u \in S, v \in T]$. That is, $\langle f, Kg \rangle$ is the fraction of edges between S and T .

Consider the following experiment: draw $u \sim \rho$ and do one step of a random walk to v . Then one can show that the probability distribution of v is ρK .

It is now natural to ask: what about two steps of a random walk? In other words, what is K^2 ($= K \circ K$)?

Fact: $(K^2 f)(u) = \mathbb{E}_{u \rightarrow \rightarrow w}[f(w)]$ (2 steps, or $v \sim u, w \sim v$).

Proof. $(K^2 f)(u) = (K(Kf))(u) = \mathbb{E}_{v \sim u}[Kf(v)] = \mathbb{E}_{v \sim u}[\mathbb{E}_{w \sim v}[f(w)]] = \mathbb{E}_{u \rightarrow \rightarrow w}[f(w)]$.

Corollary: For all $t \in \mathbb{N}$,

$$(K^t f)(u) = \mathbb{E}_{u \rightarrow \dots \rightarrow w \text{ in } t \text{ steps}}[f(w)].$$

Note: If $t = 0$, then $K^0 := I$ is the identity operator, that is, $(If)(u) := f(u) = \mathbb{E}_{u \rightarrow w \text{ in } 0 \text{ steps}}[f(w)]$.

Now let us come back to $\mathcal{E}[f]$ and rewrite it in terms of K :

$$\mathcal{E}[f] = \frac{1}{2} \mathbb{E}_{u \sim v}[(f(u) - f(v))^2] = \langle f, f \rangle - \mathbb{E}_{u \sim v}[f(u)f(v)] = \langle f, f \rangle - \langle f, Kf \rangle = \langle f, (I - K)f \rangle.$$

Definition 3. $L := I - K$ to be the normalized Laplacian operator. That is,

$$\mathcal{E}[f] = \langle f, Lf \rangle.$$

Consider again the case when $f = 1_S$ for some $S \subseteq V$. Recall that, in this case,

$$\langle f, Lf \rangle = \mathcal{E}[f] = \Pr_{u \sim v}[u \in S, v \notin S],$$

and $\langle f, f \rangle = \mathbb{E}_{u \sim \pi}[f(u)^2] = \Pr_{u \sim \pi}[u \in S] = \text{vol}(S)$. Then the ratio between them is

$$\frac{\mathcal{E}[f]}{\mathbb{E}_{u \sim \pi}[f(u)^2]} = \frac{\Pr_{u \sim v}[u \in S, v \notin S]}{\Pr_{u \sim \pi}[u \in S]} = \Pr_{u \sim v}[v \notin S \mid u \in S].$$

In other words, if we pick a random vertex of S and do one random step, the above probability is the probability of going out of S . We give it a name.

Definition: $\Phi(S) := \Pr_{u \sim v}[v \notin S \mid u \in S]$ is the conductance of S (for $S \neq \emptyset$).