

Lecture 13

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1 Overview

In this lecture, the new module **Graph Theory and applications** has begun and we are introduced to: Graphs- some definitions and properties and Random walks.

2 Graphs

Definition 1. A **graph** $G = (V, E)$ consists of a vertex set V and an edge set $E \subseteq \{(a, b) : a, b \in V\}$. Each edge connects its endpoints that are either one or two distinct vertices.

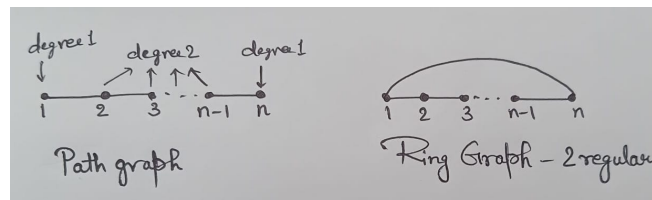
For each undirected edge (a, b) in G , there are both types of directed edges, i.e. $(a, b), (b, a) \in E$.

Definition 2. In a graph $G = (V, E)$, the **degree** of a vertex $v \in V$, i.e., $\deg(v)$ is the number of edges it touches.

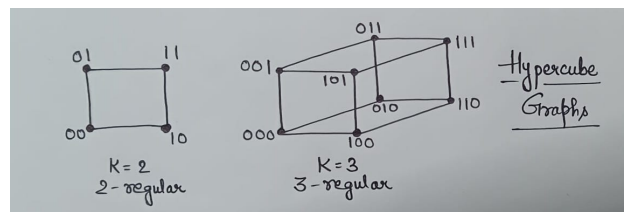
A graph is said to be **d-regular** if every vertex has degree d .

2.1 Some examples of graphs

- **Friendship Graph:** People are vertices and $\exists$ an edge between two people iff they are friends.
- **Network Graph:** Devices like computers and routers are vertices and $\exists$ an edge between the pairs of devices that are connected.
- **Path Graph:** Graph on n vertices where $V = \{1, 2, \dots, n\}$ and $E = \{(i, i+1) : i \in [n-1]\}$.



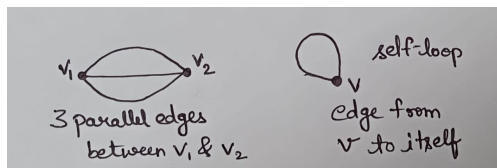
- **Ring Graph:** Graph on n vertices where $V = \{1, 2, \dots, n\}$ and $E = \{(i, i+1) : i \in [n-1]\} \cup (1, n)$.



- **Hypercube Graph:** Graph on 2^k vertices where $V = \{0, 1\}^k$ and the edges exist between the vertices differing in exactly one bit.

2.2 Certain assumptions about the graphs to be studied in this module:

- The graphs are *finite*, i.e., V, E are finite.
- In the graphs, multiple parallel edges are allowed but not self-loops.
In the below figure, $\deg(v_1) = \deg(v_2) = 3$ and $\deg(v) = 1$.



- The graphs containing *isolated* vertices, i.e., vertices of degree 0 are not allowed.

2.3 Labeling Functions

Our interest lies in those graphs where the vertices are labeled by real numbers, i.e., the functions $f : V \rightarrow \mathbb{R}$.

2.3.1 Examples

- If the graph represents physical objects, then f can be interpreted as the temperature.
- If the graph represents electrical network, then f can be interpreted as the voltage.
- For $S \subseteq V$ in a graph, f can be the $(0, 1)$ -indicator function of S , i.e., $\forall v \in V$

$$f(v) = \mathbb{1}_S = \begin{cases} 1, & v \in S \\ 0, & v \notin S \end{cases}$$

2.3.2 Representation of f as a Vector

Since $V = \{v_1, v_2, \dots, v_n\}$, the function $f : V \rightarrow \mathbb{R}$ can be represented as a column vector:

$$\begin{bmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{bmatrix} \equiv f : V \rightarrow \mathbb{R}.$$

Exercise 3. Show that $F = \{f | f : V \rightarrow \mathbb{R}\}$ with $|V| = n$ is an n -dimensional vector space (with pointwise addition and scalar multiplication) over $\mathbb{R}$.

Proof. 1. **Closure under Addition and Scalar Multiplication :** For any $f, g \in F$ and scalar $\alpha \in \mathbb{R}$, we have :

$$(f + g)(v) = f(v) + g(v) \in \mathbb{R}, \quad \forall v \in V$$

$\implies f + g \in F$, proving F is closed under addition.

$$(\alpha f)(v) = \alpha f(v) \in \mathbb{R}, \quad \forall v \in V$$

$\implies \alpha f \in F$, proving F is closed under scalar multiplication.

2. **Existence of Additive Identity** : Define the zero function $f_0(\in F) : V \rightarrow \mathbb{R}$ (additive identity) by :

$$f_0(v) = 0, \quad \forall v \in V$$

Then

$$(f + f_0)(v) = f(v) + f_0(v) = f(v) + 0 = f(v) = (f_0 + f)(v), \quad \forall f \in F$$

3. **Existence of Additive Inverse** : For each $f \in F$, define $-f \in F$ by :

$$(-f)(v) = -f(v), \quad \forall v \in V$$

Then

$$(f + (-f))(v) = f(v) + (-f(v)) = 0 = f_0(v) = (-f + f)(v)$$

$\implies -f$ is the additive inverse of $f \quad \forall f \in F$.

4. **Commutativity of Addition** : For any $f, g \in F$, we have

$$(f + g)(v) = f(v) + g(v) = g(v) + f(v) = (g + f)(v), \quad \forall v \in V$$

Thus, F is commutative under addition.

5. **Associative Properties** : For all $f, g, h \in F$ and $\alpha, \beta \in \mathbb{R}$, we have :

$$(f + g) + h = f + (g + h)$$

and

$$\alpha(\beta f) = (\alpha\beta)f$$

6. **Distributive Properties** : For all $\alpha, \beta \in \mathbb{R}$ and $f, g \in F$:

$$\alpha(f + g) = \alpha f + \alpha g$$

and

$$(\alpha + \beta)f = \alpha f + \beta f$$

7. **Multiplicative Identity** : $\forall f \in F$, we have

$$(1 \cdot f)(v) = 1 \cdot f(v) = f(v), \quad \forall v \in V, \quad 1 \in \mathbb{R}$$

Since all vector space axioms are satisfied by the property of real numbers, F is a vector space over $\mathbb{R}$.

Determining the Dimension of F : Since $|V| = n$, we can represent each function $f : V \rightarrow \mathbb{R}$ as an ordered n -tuple:

$$(f(v_1), f(v_2), \dots, f(v_n)) \in \mathbb{R}^n.$$

Define the basis of F as the set of linearly independent functions $\{e_1, e_2, \dots, e_n\}$, where:

$$e_i(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Each function $f \in F$ can be uniquely written as:

$$f = f(v_1)e_1 + f(v_2)e_2 + \dots + f(v_n)e_n$$

Since the basis consists of n elements, the vector space F has dimension n . □

3 Dirichlet Form or Local Variance or Quadratic form of the Laplacian

This is the key to Spectral graph theory.

Definition 4. The *Dirichlet form* $\mathcal{E}(f)$ for a function $f : V \rightarrow \mathbb{R}$ on a graph $G = (V, E)$ is defined as:

$$\mathcal{E}(f) := \frac{1}{2} \mathbb{E}_{u \sim v} [(f(u) - f(v))^2],$$

where $u \sim v$ means uniform random edge $(u, v) \in E$.

i.e., $\mathcal{E}(f)$ is the average squared value of f 's values along the edges.

3.1 Some properties of $\mathcal{E}(f)$

Exercise 5. Prove the following properties.

1. $\mathcal{E}(f) \geq 0$

Proof. Since squares are always non-negative:

$$(f(u) - f(v))^2 \geq 0, \quad \forall (u, v) \in E$$

and the expectation of a non-negative quantity is also non-negative, so

$$\implies \mathcal{E}(f) \geq 0.$$

□

2. $\mathcal{E}(c \cdot f) = c^2 \mathcal{E}(f)$, where c is a constant.

Proof.

$$\begin{aligned} \mathcal{E}(c \cdot f) &= \frac{1}{2} \mathbb{E}_{u \sim v} [(c \cdot f(u) - c \cdot f(v))^2] \\ &= \frac{1}{2} \mathbb{E}_{u \sim v} [c^2 (f(u) - f(v))^2] \end{aligned}$$

Since c^2 is a constant,

$$\begin{aligned} \mathcal{E}(c \cdot f) &= c^2 \frac{1}{2} \mathbb{E}_{u \sim v} [(f(u) - f(v))^2] \\ &\implies \mathcal{E}(c \cdot f) = c^2 \mathcal{E}(f) \end{aligned}$$

□

3. $\mathcal{E}(c + f) = \mathcal{E}(f)$, where c is a constant.

Proof.

$$\begin{aligned} \mathcal{E}(c + f) &= \frac{1}{2} \mathbb{E}_{u \sim v} [((c + f)(u) - (c + f)(v))^2] \\ &= \frac{1}{2} \mathbb{E}_{u \sim v} [(c + f(u) - c - f(v))^2] \\ &= \frac{1}{2} \mathbb{E}_{u \sim v} [(f(u) - f(v))^2] \\ &\implies \mathcal{E}(c + f) = \mathcal{E}(f) \end{aligned}$$

□

Intuition: $\mathcal{E}(f)$ is small $\iff f$'s values don't vary much along edges.

Example 6. Let $S \subseteq V$ and $f = \mathbb{1}_S$, then

$$\begin{aligned}\mathcal{E}(f) &= \frac{1}{2} \mathbb{E}_{u \sim v} [(\mathbb{1}_S(u) - \mathbb{1}_S(v))^2] \\ &= \frac{1}{2} \mathbb{E}_{u \sim v} [\mathbb{1}(u \text{ and } v \text{ are on opposite sides of the partition } S \text{ and } V \setminus S)] \\ &= \frac{1}{2} \mathbb{E}_{u \sim v} [\mathbb{1}((u,v) \text{ crosses the cut } (S, V \setminus S))] \\ &= \frac{1}{2} (\text{fraction of edges on } \partial S), \quad \text{where } \partial S \text{ is the boundary of } S\end{aligned}$$



4 Random Vertices and Random Walks

4.1 Choosing a Random Vertex

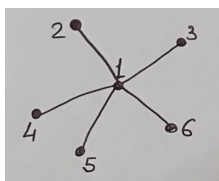
We can't just pick a random v from V unless the graph is regular. The correct method:

1. Pick a random edge $(u, v) \in E$.
2. Output u (the initial vertex).

Let's denote this probability distribution over V by π .

π is biased towards choosing the vertices with high degree because they touch the most of the edges.

Example 7. Consider this graph, where 1 is the center vertex and rest are leaves. The total number of directed edges is 10.



To get 1 as the output, we could have picked $(1, v)$ or $(v, 1)$, $\forall v \in \{2, \dots, 6\}$ each with probability $\frac{1}{2}$ but only $(1, v)$ gives 1, thus

$$\pi(1) = \frac{5}{10} = \frac{1}{2}$$

Similarly, only $(v, 1)$ gives v , $\forall v \in \{2, \dots, 6\}$, thus

$$\pi(v) = \frac{1}{10}, \quad v \in \{2, 3, 4, 5, 6\}$$

Exercise 8. Show that $\pi(u) = \frac{\deg(u)}{2|E|}$.

Proof. We define the distribution $\pi(u)$ as the probability that a vertex $u \in V$ is the output of a uniformly randomly chosen edge (u, v) .

The number of directed edges that outputs $u = \deg(u)$ in the original undirected graph.

Since we are drawing a uniformly random edge, i.e., we are drawing a uniformly random directed edge and there are $2|E|$ directed edges, so the probability that its initial vertex is u :

$$\pi(u) = \frac{\deg(u)}{2|E|}$$

□

- Since $\pi(u)$ is proportional to the degree, vertices with higher degree have higher probability of being chosen. So, the probability of choosing any vertex is not equally likely, i.e., π is not a uniform distribution on V .
- If G is a regular graph, then $\pi \equiv$ uniform probability distribution on V .

Exercise 9. Prove that the following procedure: pick $u \sim \pi$ and then pick v as a uniform random neighbor of u (1- step random walk) is equivalent to drawing a uniformly random edge (u, v) .

That is, picking $u \sim \pi$ and then picking v as a uniform random neighbor of u , is distributed according to π , or

pick a uniform random edge (u, v) , whether the output is u or v , both give π .

Proof. The given procedure can be described as a way of choosing a pair of vertices u and v where first u is drawn from π , so the probability of u being chosen $= \frac{\deg(u)}{2|E|}$.

Then choosing a uniformly random neighbor of u , say v . Now, $|N(u)| = \deg(u)$ and each of them is chosen with equal probability, which is $\frac{1}{\deg(u)}$.

So, the probability that u and v are chosen together is

$$\frac{\deg(u)}{2|E|} \cdot \frac{1}{\deg(u)} = \frac{1}{2|E|},$$

which is the probability of choosing a particular directed edge (u, v) .

Thus, if we pick $u \sim \pi$ and let v be its uniformly random neighbor then (u, v) is a uniformly random directed edge.

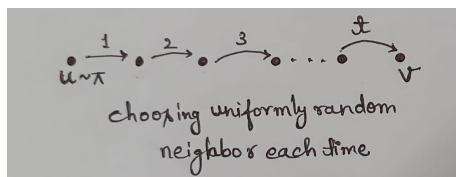
After those two steps if we output its terminus we get v , then the distribution on v is given by $\frac{\deg(v)}{2|E|}$ which is equal to $\pi(v)$.

So v can be thought of as being drawn by first choosing a directed edge and then getting its terminus as output. Thus, if we chose v to be the output in the probability distribution earlier (4.1) it would have been the same probability distribution π .

□

4.2 Corollary

Let $t \in \mathbb{N}$. Pick $u \sim \pi$. Now do a random walk starting at u and taking t steps: Then the distribution of the



endpoint of the walk, v is π , i.e., $v \sim \pi$.

Definition 10. π is the **invariant/stationary** distribution on vertices for the random walks in graphs.

Question: If $u_0 \in V$ is not chosen from π and we start a random walk from u_0 , then as $t \rightarrow \infty$, does the distribution of $v \rightarrow \pi$ where v is the final vertex where walk ends?

Answer:

1. No, if G is disconnected as the walk cannot reach the vertices in other connected components.

In the case of a disconnected regular graph also, suppose we start from a vertex in one component, say C_1 and do a random walk, then we will always be in C_1 . Therefore, no matter how large t is, the distribution on v won't be uniform on the whole graph (like π) but only on C_1 where we started.

2. No, if G is bipartite.

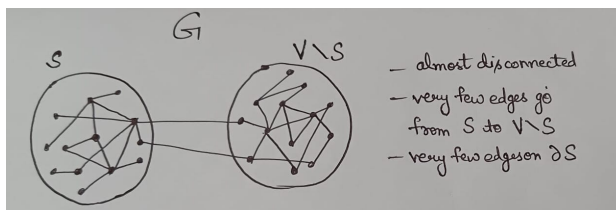
In G , there are two partitions, say X (on the left) and Y (on the right) and we start from a vertex in X . If t is odd then v will always be in Y whereas if t is even then v will always be in X .

In both the cases, v will have a distribution that is uniformly distributed on Y or X vertices but not on all vertices which is the stationary distribution π . Same will happen in case of regular graphs too.

3. Otherwise, yes.

Question: How large should t be to get ε -close to π by doing a random walk if G is connected and not bipartite?

Example: Consider an almost disconnected graph: Let G be a graph with two large components, $S \subset V$



and $S^c = V \setminus S$ connected by only a few edges.

If we take regular graphs, then ∂S will have few edges that will block the random walk from getting to uniform random distribution on all vertices in short time.

Recall : $\mathcal{E}(\mathbb{1}_S)$ gives fraction of edges on ∂S . Thus, if $\mathcal{E}(\mathbb{1}_S)$ is small, it prevents fast convergence to limiting distribution.

In the upcoming lectures, we will see that fast convergence $\iff \mathcal{E}(f)$ is never small.

5 Mean and Variance

Let $f : V \rightarrow \mathbb{R}$ and $u \sim \pi$, then $f(u)$ is a real-valued random variable and its mean and variance are defined as :

- **Mean :**

$$\mathbb{E}[f] := \mathbb{E}_{u \sim \pi}[f(u)] = \mu$$

- **Variance :**

$$\text{Var}[f] := \mathbb{E}_{u \sim \pi}[(f(u) - \mu)^2]$$

Exercise 11. Prove that

$$\text{Var}[f] = \frac{1}{2} \mathbb{E}_{\substack{u \sim \pi \\ v \sim \pi \\ \text{independent}}} [(f(u) - f(v))^2]$$

Proof.

$$\begin{aligned}\text{Var}[f] &= \mathbb{E}_{u \sim \pi}[(f(u) - \mu)^2] \\ &= \mathbb{E}_{u \sim \pi}[f(u)^2 - 2\mu f(u) + \mu^2]\end{aligned}$$

Using the linearity of expectation,

$$\begin{aligned}\text{Var}[f] &= \mathbb{E}_{u \sim \pi}[f(u)^2] - 2\mu \mathbb{E}_{u \sim \pi}[f(u)] + \mu^2 \\ &= \mathbb{E}_{u \sim \pi}[f(u)^2] - 2\mu \cdot \mu + \mu^2 \\ &= \mathbb{E}_{u \sim \pi}[f(u)^2] - \mu^2\end{aligned}$$

Now, consider two independent samples $u, v \sim \pi$:

$$\begin{aligned}\mathbb{E}_{\substack{u \sim \pi \\ v \sim \pi}}[(f(u) - f(v))^2] &= \mathbb{E}_{\substack{u \sim \pi \\ v \sim \pi}}[f(u)^2 - 2f(u)f(v) + f(v)^2] \\ &= \mathbb{E}_{u \sim \pi}[f(u)^2] - 2\mathbb{E}_{\substack{u \sim \pi \\ v \sim \pi}}[f(u)f(v)] + \mathbb{E}_{v \sim \pi}[f(v)^2] \\ &= 2\mathbb{E}_{u \sim \pi}[f(u)^2] - 2\mathbb{E}_{\substack{u \sim \pi \\ v \sim \pi}}[f(u)f(v)]\end{aligned}$$

Since u, v are independent, their expectations factorize:

$$\mathbb{E}_{\substack{u \sim \pi \\ v \sim \pi}}[f(u)f(v)] = \mathbb{E}_{u \sim \pi}[f(u)]\mathbb{E}_{v \sim \pi}[f(v)] = \mu^2.$$

Thus,

$$\begin{aligned}\mathbb{E}_{\substack{u \sim \pi \\ v \sim \pi}}[(f(u) - f(v))^2] &= 2\mathbb{E}_{u \sim \pi}[f(u)^2] - 2\mu^2 = 2 \cdot \text{Var}[f] \\ \implies \text{Var}[f] &= \frac{1}{2} \mathbb{E}_{\substack{u \sim \pi \\ v \sim \pi \\ \text{independent}}}[(f(u) - f(v))^2]\end{aligned}$$

The local variance $\mathcal{E}(f)$ evaluated the squared quantity for a random edge, thus $\text{Var}[f]$ computed above represents global variance. $\square$

Remark : Later on, we will study expander graphs for which $\mathcal{E}(f)$ is large as compared to $\text{Var}[f]$.

Example 12. Let $S \subseteq V$ and $f = \mathbb{1}_S$, then

- **Expectation of f :**

$$\mathbb{E}[f] = \mu = \mathbb{P}_{u \sim \pi}[u \in S] = \text{vol}(S), \text{ where } \text{vol}(S) \text{ is the volume or weight of } S$$

- **Variance of f :**

$$\begin{aligned}\text{Var}[f] &= \mathbb{E}[\mathbb{1}_S^2] - \mu^2 \\ &= \mu - \mu^2, \quad \text{as } \mathbb{E}[\mathbb{1}_S^2] = \mathbb{E}[\mathbb{1}_S] = \mu \\ &= \mu(1 - \mu) \\ &= \text{vol}(S) \cdot (1 - \text{vol}(S)) \\ &= \text{vol}(S) \cdot \text{vol}(S^c), \quad \text{where } S^c = V \setminus S\end{aligned}$$

- **Dirichlet Form of f :** Since $\mathcal{E}(f)$ captures the fraction of edges crossing the cut $(S, V \setminus S)$, so

$$\mathcal{E}(f) = \mathbb{P}_{u \sim v}[u \in S, v \notin S]$$

Further, we will study how the fraction of edges in ∂S compares to $\text{vol}(S)$.

References

- Lecture 13 notes
- Ryan O'Donnell's course on CS Theory Toolkit