

Assignment 3

Discrete Fourier Analysis

Instructor: Sruthi Sekar

TAs: Krishna N Agaram, Nilabha Saha

§ 1 Fourier Expansions

Compute the Fourier expansions, the mean, and the variance of the following functions:

- The inner product modulo 2 function, $\text{IP}_{2n} : \mathbb{F}_2^{2n} \rightarrow \{-1, 1\}$ defined by $\text{IP}_{2n}(x_1, \dots, x_n, y_1, \dots, y_n) = (-1)^{x \cdot y}$, where $x \cdot y = \sum_{i=1}^n x_i y_i$.
- $\text{OXR} : \mathbb{F}_2^3 \rightarrow \{0, 1\}$, defined by $\text{OXR}(x_1, x_2, x_3) = x_1 \vee (x_2 \oplus x_3)$. Here, $\vee$ denotes logical OR, and $\oplus$ denotes logical XOR.
- The density function $\varphi_{\{a, a+e_i\}} : \mathbb{F}_2^n \rightarrow \mathbb{R}^{\geq 0}$, where $a \in \mathbb{F}_2^n$ and $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ with the 1 in the i -th coordinate. (Ref: definition of density function in [lecture 8](#))

§ 2 Properties of Fourier

- Let $f : \mathbb{F}_2^n \rightarrow \{0, 1\}$, $n > 1$, and suppose that $\#\{x : f(x) = 1\}$ is odd. Prove that all of f 's Fourier coefficients are non-zero.
- Prove that any $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ has at most one Fourier coefficient with magnitude exceeding $1/2$. Is this also true for any $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ with $\|f\|_2 = 1$? (Notation: Norm of f is defined as $\|f\|_2 := \sqrt{\langle f, f \rangle}$)
- Let $F : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be a random function (i.e., each $F(x)$ is ± 1 with probability $1/2$ each, independently for all $x \in \{-1, 1\}^n$). Show that for each $S \subseteq [n]$, the random variable $\hat{F}(S)$ has mean 0 and variance 2^{-n} .

§ 3 Boolean Dual

The Boolean dual of $f : \{-1, +1\}^n \rightarrow \mathbb{R}$ is the function $f^\dagger$ defined by $f^\dagger(x) = -f(-x)$. The function f is said to be odd if it equals its dual; equivalently, if $f(-x) = -f(x)$ for all x . The function f is said to be even if $f(-x) = f(x)$ for all x . Given any function $f : \{-1, +1\}^n \rightarrow \mathbb{R}$, its odd part is the function $f^{\text{odd}} : \{-1, +1\}^n \rightarrow \mathbb{R}$ defined by $f^{\text{odd}}(x) = (f(x) - f(-x))/2$, and its even part is the function $f^{\text{even}} : \{-1, +1\}^n \rightarrow \mathbb{R}$ defined by $f^{\text{even}}(x) = (f(x) + f(-x))/2$.

- Express $\hat{f}^\dagger(S)$ in terms of $\hat{f}(S)$.
- Verify that $f = f^{\text{odd}} + f^{\text{even}}$ and that f is odd (respectively, even) if and only if $f = f^{\text{odd}}$ (respectively, $f = f^{\text{even}}$).

c) Show that

$$f^{\text{odd}} = \sum_{\substack{S \subseteq [n] \\ |S| \text{ odd}}} \hat{f}(S) \chi_S, \quad f^{\text{even}} = \sum_{\substack{S \subseteq [n] \\ |S| \text{ even}}} \hat{f}(S) \chi_S$$

§ 4 Affine Testing

- a) We call $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ an affine function if $f(x) = a \cdot x + b$ for some $a \in \mathbb{F}_2^n$, $b \in \mathbb{F}_2$. Show that f is affine if and only if $f(x) + f(y) + f(z) = f(x + y + z)$ for all $x, y, z \in \mathbb{F}_2^n$.
- b) Let $f : \mathbb{F}_2^n \rightarrow \mathbb{R}$. Suppose that we choose $X, Y, Z \sim \mathbb{F}_2^n$ independently and uniformly. Show that $\mathbb{E}[f(X)f(Y)f(Z)f(X + Y + Z)] = \sum_S \hat{f}(S)^4$.
- c) Give a 4-query test for a function $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ with the following property: if the test accepts with probability $1 - \varepsilon$ then f is ε -close to being affine.