

Lecture 3

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1 Welcome to Lecture 3 of CS 785

1.1 Lecture Overview

- Stirling's Approximation for $n!$
- Applications of Stirling's Approximation formula

1.2 References

- "A History of Probability and Statistics and Their Applications before 1750" by Anders Hald
- Ryan O'Donnell's course on CS Theory Toolkit

2 Stirling's Approximation for $n!$

Let us begin with an example.

Example 1. What is the probability that flipping a fair coin $2n$ times results in exactly n heads and n tails? The answer is $\binom{2n}{n} \frac{1}{2}^{2n}$. Here we have $\binom{2n}{n}$ as a binomial coefficient which we aim to approximate. We would like to see how the probability computed can be expressed as a function of n .

2.1 De Moivre's Approximation

The French Mathematician De Moivre came up with the following: $\binom{2n}{n} \frac{1}{2}^{2n} \approx \frac{2.168(1-\frac{1}{2n})^{2n}}{\sqrt{2n-1}} \approx \frac{2.168}{\sqrt{2ne}}$

Exercise 2. Prove that $\lim_{n \rightarrow \infty} (1 - \frac{1}{2n})^{2n} = \frac{1}{e}$

$$\begin{aligned}
 &\text{Proof. } \lim_{n \rightarrow \infty} (1 - \frac{1}{2n})^{2n} \\
 &= \lim_{n \rightarrow \infty} e^{\ln (1 - \frac{1}{2n})^{2n}} \\
 &= \lim_{n \rightarrow \infty} e^{2n \ln (1 - \frac{1}{2n})} \\
 &= e^{\lim_{n \rightarrow \infty} 2n \ln (1 - \frac{1}{2n})} \\
 &= e^{\lim_{n \rightarrow \infty} \frac{\ln (1 - \frac{1}{2n})}{\frac{1}{2n}}} \\
 &= e^{\lim_{n \rightarrow \infty} \frac{\frac{1}{2n^2} (1 - \frac{1}{2n})}{\frac{-1}{2n^2}}} \quad [\text{Applying L'Hôpital's rule}] \\
 &= \frac{1}{e} \quad [\text{Proved}]
 \end{aligned}$$

□

Now it can be understood that $\frac{2.168(1-\frac{1}{2n})^{2n}}{\sqrt{2n-1}} \approx \frac{2.168}{\sqrt{2n(1-\frac{1}{2n})e}} \approx \frac{2.168}{\sqrt{2ne}}$

Stirling was trying to approximate $\ln(n!)$

Our Goal for this lecture: Find $f(n)$ such that $n! \sim f(n)$

2.2 Attempt I in Approximation of n!

Recap: $f(n) \sim g(n) \implies \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$

Now $n! = n(n-1)(n-2) \dots 2.1$

$\implies n! \leq n.n.n \dots n$

$\implies n! \leq n^n \dots \text{Eq(1)}$

Again, $n! = n(n-1)(n-2) \dots 2.1$

We drop the lower half of terms to find the lower bound.

$n! \geq \frac{n}{2} \frac{n}{2} \frac{n}{2} \dots (\frac{n}{2} \text{ terms})$

$\implies n! \geq \frac{n}{2}^{\frac{n}{2}} \dots \text{Eq(2)}$

From Eq(1) and Eq(2): $\frac{n}{2}^{\frac{n}{2}} \leq n! \leq n^n$

Taking logs to understand better: $\frac{n}{2} \log_2(\frac{n}{2}) \leq \log_2(n!) \leq n \log_2(n)$

$\therefore \frac{1}{2} n \log_2(n) - \theta(n) \leq \log_2(n!) \leq n \log_2(n)$

We see that $\frac{1}{2} n \log_2(n) - \theta(n)$ & $n \log_2(n)$ are factor $\frac{1}{2}$ apart

$\therefore \log_2(n!) = \theta(n \log_2(n))$

$\implies n! = 2^{\theta(n \log_2(n))}$

Looks like a good start, but we can do better!

2.3 Attempt II in Approximation of n! - using Taylor Series

Taylor Series Definition of e^x :

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$\implies e^x \geq \frac{x^n}{n!} \forall x \in \mathbb{R}^+$

$\implies n! \geq \frac{x^n}{e^x} \forall x \in \mathbb{R}^+ \dots \text{Eq(3)}$

Treating n as constant we now find out the maxima of $\frac{x^n}{e^x}$

$\frac{d}{dx} \left(\frac{x^n}{e^x} \right) = 0$

$\implies e^x n x^{n-1} - x^n e^x = 0$

$\implies n - x = 0$

Now $\frac{d^2}{dx^2} \left(\frac{x^n}{e^x} \right) = \frac{d}{dx} \left(\frac{e^x n x^{n-1} - x^n e^x}{e^{2x}} \right) = \frac{d}{dx} \left(\frac{n x^{n-1} - x^n}{e^x} \right) = \frac{n(n-1)x^{n-2} - 2n x^{n-1} + x^n}{e^x} = \frac{-n^{n-1}}{e^n} < 0$ at $x=n$

Thus maxima of $\frac{x^n}{e^x}$ occurs at $x=n$

Putting $x=n$ in Eq(3) gives us $n! \geq \frac{n^n}{e^n}$

$\implies \ln n! \geq n \ln \frac{n}{e}$

$\implies \ln n! \geq n \ln n - n$

In **Attempt I** we derived $n! \leq n^n \implies \ln n! \leq n \ln n$

Comparing our results so far we obtain:

$\ln n! \sim n \ln n$ [treating $-n$ as the error term]

The question we face now is which one of $\frac{n^n}{e^n}$ and n^n is closer to $n!$

2.4 Attempt III in Approximation of n! - using Integral tricks

$\ln n! = \ln(n(n-1)(n-2) \dots 2.1) = \ln n + \ln(n-1) + \ln(n-2) + \dots + \ln 2 + \ln 1$

From Figure 1 we can see

$\ln n! = \text{Sum of areas of the rectangles} \geq \text{Area under the curve } \widetilde{AB} = \int_1^n \ln x dx = n \ln n - n + 1$

$\therefore \ln n! \geq n \ln n - n + 1 \dots \text{Eq(4)}$

From Figure 1 we also see

$\ln n! = \text{Area under the curve } \widetilde{AB} + \text{Area of shaded region of curved triangles}$

Now area of shaded region of curved triangles $\leq$ Sum of the areas of the straightened triangles $= \frac{1}{2} \ln n$

$\therefore \ln n! \leq n \ln n - n + 1 + \frac{1}{2} \ln n \dots \text{Eq(5)}$

Comparing Eq(4) & Eq(5) we have:

$n \ln n - n + 1 \leq \ln n! \leq n \ln n - n + 1 + \frac{1}{2} \ln n$

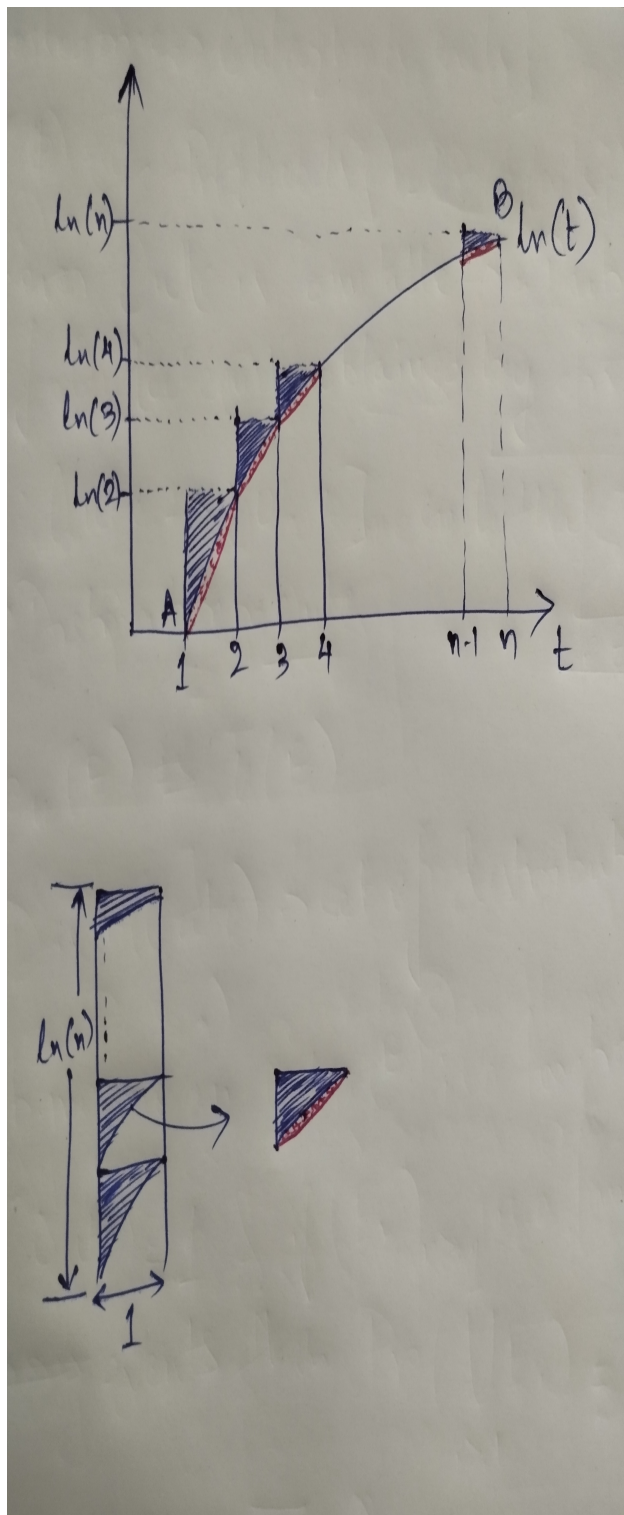


Figure 1: Graph of $\ln n$

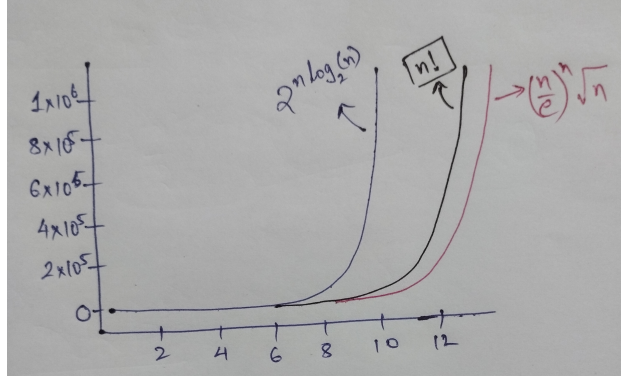


Figure 2: Comparisons of different bounds

$$\Rightarrow e\left(\frac{n}{e}\right)^n < n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$$

$$\therefore n! \sim \theta\left(\frac{n^n}{e^n}\right)$$

Exercise 3. Show that the area of the Red portion in the graph of $\ln n$ in Figure 1 is $O(1)$

Note: Sruthi Ma'am would provide the proof later.

This improves the lower bound of $n!$:-

$$\ln n! \geq n \ln n - n + 1 - O(1)$$

$$\Rightarrow n! = \theta\left(\frac{n^n}{e^n} \sqrt{n}\right) \dots \text{Eq(6)}$$

This is extremely close to what Stirling's formula shows us: $n! \sim \sqrt{2\pi} \sqrt{n} \left(\frac{n}{e}\right)^n$

Thus the constant hidden by θ in Eq(6) is $\sqrt{2\pi}$ which is the actual area of the Red shaded region in Figure 1

Example 4. $10! = 3628800$

Using Stirling's approximation of $n!$ we obtain: $10! \sim \sqrt{2\pi} \sqrt{10} \left(\frac{10}{e}\right)^{10} = 3598695.61$ which gives us less than 1% error

2.5 Plots for different bounds obtained so far

Figure 2 shows the plots for the different bounds obtained so far

3 Applications of Stirling's formula

- Binomial Coefficients
- Minimum no. of comparisons to sort n elements using any comparison-based sorting algorithm

3.1 Binomial Coefficients

Going back to our first example of this lecture:

Probability of the occurrence of exactly n heads in $2n$ tosses of a fair coin

$$= \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} = \frac{2n!}{n!n!} \left(\frac{1}{2}\right)^{2n} \sim \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi 2n}}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \left(\frac{1}{2}\right)^{2n} = \frac{1}{\sqrt{n\pi}}$$

Now to estimate $\binom{n}{k}$ there is case-by-case approximation analysis based on the value of k .

We are only concerned about the case where $k = \theta(n)$

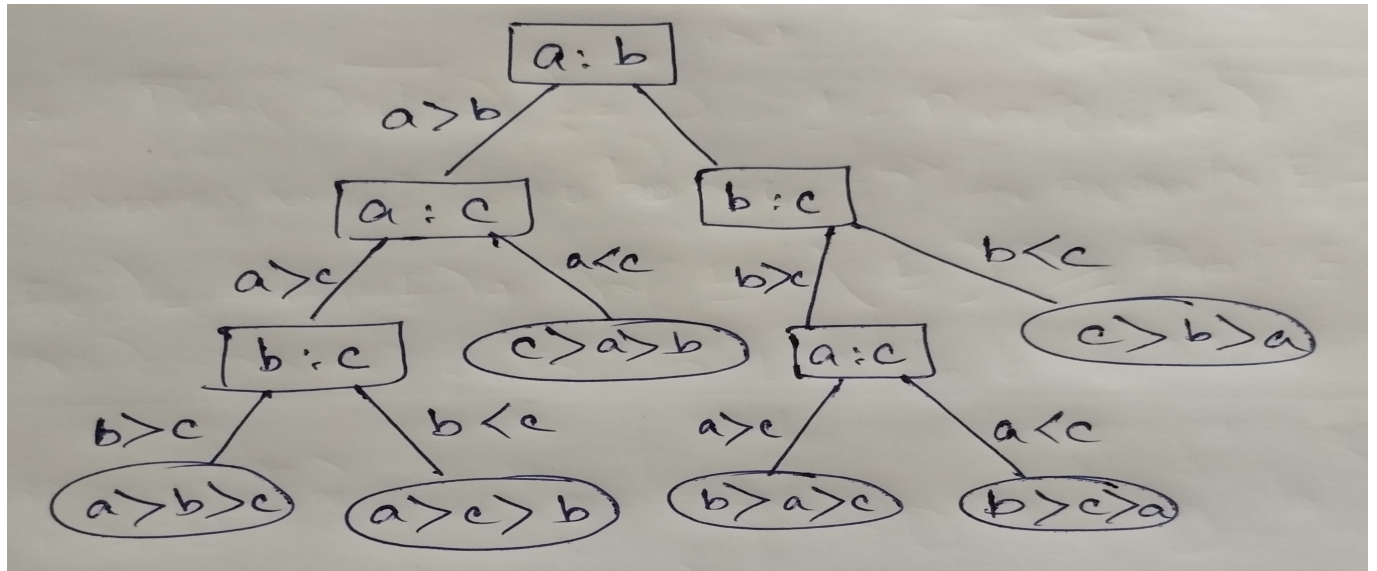
$\therefore \binom{n}{k} = \binom{n}{n-k}$ we will consider $k = pn$, $0 < p \leq \frac{1}{2}$ as it suffices for $k \leq \frac{n}{2}$

Let $q = (1-p)$

$\therefore \binom{n}{k} = \binom{n}{pn} = \frac{n!}{(pn)!(qn)!} \sim \frac{(\frac{n}{e})^n \sqrt{2\pi n}}{(\frac{pn}{e})^{pn} \sqrt{2\pi pn} (\frac{qn}{e})^{qn} \sqrt{2\pi qn}} = \frac{1}{\sqrt{2\pi pq}} \frac{1}{\sqrt{n}} [(\frac{1}{p})^p (\frac{1}{q})^q]^n$
 Later in this course we will see Binary Entropy function of $p = 2^{H_2(p)} = (\frac{1}{p})^p (\frac{1}{q})^q$

3.2 Minimum no. of comparisons to sort n elements using any comparison-based sorting algorithm

Exercise 5. Show that the minimum no. of comparisons to sort n elements using any comparison-based sorting algorithm is $\Omega(n \log_2 n)$



Proof. We use a decision tree in the figure above to represent the comparisons made by a sorting algorithm on an input array of 3 elements.

The internal node "a:b" represents the comparison between only the elements a & b. So a decision is to be made on the basis of this comparison. This node has two edges based on the two possibilities. In this way the entire tree covers all possible permutations of 3 elements. We can see all the $3!=6$ permutations occur in all the leaves of this tree.

It is absolutely correct to say that a decision tree used to represent the comparisons made to sort an input array of n elements will have all the $n!$ permutations in its leaves.

Thus the binary tree has at least $n!$ leaves.

We know a binary tree of height h has maximum no. of leaves $\leq 2^h$ [$\because$ at every level i there are maximum 2^i nodes]

We can hence say, for L being the no. of leaves of our decision tree to represent the comparisons for sorting n elements:

$$n! \leq L \leq 2^h$$

$$\Rightarrow n! \leq 2^h$$

$$\Rightarrow \log_2 n! \leq \log_2 2^h$$

$$\Rightarrow h \geq \log_2 n! \sim n \log_2 \left(\frac{n}{e}\right) + \log_2 \sqrt{2\pi n} = \Omega(n \log_2 n)$$

$$\Rightarrow h \geq \Omega(n \log_2 n)$$

Thus we have shown that the height of the decision tree representing the comparisons to sort n elements is at least $\Omega(n \log_2 n)$

The height of the tree is the length of the path from the root node to the deepest leaf node which is the leaf node at the last level of the tree. This essentially gives us the minimum no. of comparisons for a sorting algorithm to be $\Omega(n \log_2 n)$ $\square$