

Assignment 1

Asymptopia

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§ 1 Asymptotic Simplifications

In this problem, your solutions should begin with a displayed equation of the following form:

$$[\text{expression to be analyzed}] = \Theta(f(x)),$$

where $f(x)$ is a "simple" function, and x is the asymptotic parameter (either $n \rightarrow \infty$ or $\varepsilon \rightarrow 0^+$). You should then give an explanation of how you derived your answer (but a completely formal proof is not required). We give a sample problem/solution.

Sample problem. Asymptotically simplify

$$\sqrt{1 + \varepsilon} - 1.$$

Sample solution.

$$\sqrt{1 + \varepsilon} - 1 = \Theta(\varepsilon).$$

The upper bound is immediate from $\sqrt{1 + \varepsilon} \leq 1 + \varepsilon$. The lower bound follows from the fact that $\sqrt{1 + \varepsilon} \geq 1 + \varepsilon/3$ for $0 \leq \varepsilon \leq 3$, which can be proved as follows: $(1 + \varepsilon/3)^2 = 1 + \frac{2}{3}\varepsilon + \frac{1}{9}\varepsilon^2 \leq 1 + \varepsilon$, where we used $\varepsilon \leq 3 \implies \frac{1}{9}\varepsilon^2 \leq \frac{1}{3}\varepsilon$.

(a) Asymptotically simplify

$$\sqrt{n+1} - \sqrt{n}.$$

(b) Asymptotically simplify

$$\sum_{i=1}^n i^3.$$

(c) Asymptotically simplify

$$\frac{1}{1 + \varepsilon} - (1 - \varepsilon).$$

(d) Noting carefully the range of summation, asymptotically simplify

$$\sum_{i=n+1}^{n^2} \frac{1}{i}.$$

(e) Asymptotically simplify

$$\log_2 \left(\frac{1}{\frac{1}{2} - \varepsilon} \right).$$

For this problem, your solution should be a degree-1 polynomial in ε with an error of $\pm\Theta(\varepsilon^2)$. You should treat $\varepsilon \rightarrow 0$ with ε potentially positive or negative.

(f) Assume $m \geq 1$ is the solution of $m^m = n$. Asymptotically express m as a function of n .

§ 2 Bounded Differences in Sequences

There is a sequence satisfying $1 = a_1 \leq a_2 \leq \dots \leq a_n$ and $a_{i+1} - a_i \leq \sqrt{a_i}$ for all $1 \leq i < n$. Prove that

$$\sum_{i=1}^{n-1} \frac{a_{i+1} - a_i}{a_i} \leq O(\log n).$$

§ 3 A Subtle Change

Two functions $f(n)$ and $g(n)$ are said to have the *same rate of growth*, denoted as $f(n) \asymp g(n)$, if they satisfy the following definition:

$$f(n) \asymp g(n) \iff |f(n)| \leq C |g(n)| \text{ and } |g(n)| \leq C |f(n)|$$

for some C and for all sufficiently large n .

There's a stronger relation: $f(n)$ is said to be *asymptotic to* $g(n)$, denoted by $f(n) \sim g(n)$, if they satisfy the following definition:

$$f(n) \sim g(n) \iff \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1.$$

Give an example of two functions $f(n), g(n)$ such that $f(n) \asymp g(n)$ but it is not the case that $f(n) \sim g(n)$.

§ 4 Existence of Euler-Mascheroni Constant

Consider the harmonic series whose terms are given by

$$H_n = \sum_{i=1}^n \frac{1}{i}$$

and consider the difference $\Delta_n = H_n - \ln n$. In this problem, we shall try to see if this difference converges to a constant as $n \rightarrow \infty$ and bound the value of the constant if it does. We make use of the following theorem from real analysis:

Theorem 1. If $a_1, a_2, \dots$ is an increasing (resp. decreasing) monotonic sequence of real numbers such that the sequence is bounded above (resp. below), then the sequence converges.

- a) Show that $\{\Delta_n\}_{i=1}^\infty$ is a monotonically decreasing sequence.
- b) Show that the $\{\Delta_n\}_{i=1}^\infty$ is bounded below by 0.

This establishes, by the aforementioned theorem, that Δ_n converges to constant, which we shall denote by γ . This is called the Euler-Mascheroni constant.

- c) Using results from class or otherwise, show that $0 < \gamma < 1$. (Don't just lift off the exact value of γ from class though)

§ 5 The Nature of γ [Open Problem]

This is an open problem regarding the Euler-Mascheroni constant γ .

- a) Is γ rational or irrational?
- b) Is γ algebraic or transcendental over $\mathbb{Q}$?

§ 6 Berry and Esseen: Part I

- a) Show that in Berry-Esseen Theorem, we can also conclude

$$|\Pr[S < u] - \Pr[Z < u]| \leq c\gamma.$$

Hint: You will need that $\lim_{\delta \rightarrow 0^+} \Pr[Z \leq u - \delta] = \Pr[Z \leq u]$.

- b) Deduce that if $I \subseteq \mathbb{R}$ is any interval, we can also conclude

$$|\Pr[S \in I] - \Pr[Z \in I]| \leq 2c\gamma.$$

§ 7 Berry and Esseen: Part II

Show that the assumptions $\mathbb{E}[X_i] = 0$ and $\sum_{i=1}^n \text{Var}[X_i] = 1$ in the Berry-Esseen Theorem are not restrictive, as follows. Let $X_1, \dots, X_n$ be independent random variables with finite means and variances. Let $S = \sum_{i=1}^n X_i$ and let $Z \sim \mathcal{N}(\mu, \sigma^2)$, where $\mu = \sum_{i=1}^n \mathbb{E}[X_i]$ and $\sigma^2 = \sum_{i=1}^n \text{Var}[X_i]$. Assuming $\sigma^2 > 0$, show that for all $u \in \mathbb{R}$,

$$|\Pr[S \leq u] - \Pr[Z \leq u]| \leq \frac{c\epsilon}{\sigma^3},$$

where

$$\epsilon = \sum_{i=1}^n \|X_i - \mathbb{E}[X_i]\|_3^3.$$

§ 8 Interplay of Asymptotics and Probability [Hard]

Asymptotics can be used to derive several important results in probability theory. In this problem, you will use asymptotics to derive an important theorem in classical probability.

Prove the DeMoivre-Laplace Theorem:

Theorem 2 (DeMoivre-Laplace). *Let $X \sim \text{Binomial}(n, p)$ and assume that*

$$x_n = \frac{k_n - np}{\sqrt{np(1-p)}} = o(n^{1/6})$$

where k_n are the sampled values from the binomial distribution. Let φ denote the density function of the standard normal random variable. Then there exists positive constants A , B , and C such that

$$\left| \frac{\Pr[X = k_n] \sqrt{np(1-p)}}{\varphi(x_n)} - 1 \right| \leq \frac{A}{n} + \frac{B |x_n|^3}{\sqrt{n}} + \frac{C |x_n|}{\sqrt{n}}.$$

Hint: Use Stirling's approximation and Taylor expansions.